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THE MATHEMATICS TEACHER

Volume XLV · MARCH · 1952 · Number 3

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OFFICIAL JOURNAL OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W., Washington 6, D.C.

Printed at Menasha, Wisconsin, U.S.A.

Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930. Printed in U.S.A.

THE MATHEMATICS TEACHER

Official Journal of the National Council
of Teachers of Mathematics

Devoted to the interests of mathematics teachers in Elementary and Secondary Schools, Junior Colleges and Teacher Education

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1201 Sixteenth Street, N.W., Washington 6, D.C.

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THE MATHEMATICS TEACHER

Volume XLV Number 3

Let's Forget Mental Transfer

By DAVID SKOLNIK
Central High School, Newark, N. J.

Introduction. The doctrine of mental discipline dates back to the time when the mind was regarded as a muscle that could be exercised. More recently the doctrine was challenged by stubborn facts and modified. Thorndike advanced the theory that transfer took place only between situations with identical elements. This sharply limited the principle of mental transfer. After its curtailment by "identical elements," mental transfer was further demolished by the experimental evidence of Gestalt psychologists which showed that experience is always complex, never an aggregation of elements. This has led to the view that mental development consists of the power to think abstractly and to form general ideas. This view would in fact make mental transfer inherent in the definition of mental development.

Traditionally, teachers of mathematics have had a fundamental stake in the fortunes of the doctrine of mental transfer. This is particularly true with respect to demonstrative geometry. In a number of questionnaires to teachers, notably one by Mr. Shibli, the following was selected as the most important objective for the teaching of geometry: "To develop the habit of clear thinking and precise expression."* It is evident that the doctrine of mental transfer is in a state of flux, rising and ebbing with various experimental and *a priori* developments so that the last word on it will probably never be known. The question that naturally arises is: Must teachers of mathematics continue to be bogged down in the quagmire of shifting views about mental training?

The thesis of this article is that, insofar as it relates to geometry, there is an alternative which will make us independent of the ultimate disposition of mental transfer without disclaiming the broadening values in which we believe. The alternative is to *forget mental transfer* and broaden the presentation as well as the subject matter. The proposal is not to transfer the mode of thinking from geometry to other fields but rather to present the subject so broadly in the first place that the emphasis will rest on the mode of thinking rather than on the elements to which the thinking applies. It is only incidental that this method agrees with the present status of the mental transfer theory which is embedded in the definition of mental development. The important thing is that our teaching would not depend on our ability to transfer but would be merely a series of applications to various elements, of which the mathematical would probably be the simplest.

The reasons why such a plan has not been tried on a large scale are some confused notions about the lack of time, the

* THE MATHEMATICS TEACHER, Vol. XLIII, No. 3, Mar. 1950.

traditional scope of geometry, the lack of material suitable to 15-year-olds, and the general inertia of human nature. Fortunately, many teachers are now seriously asking: How can the teaching of geometry be made so general that it will serve our pupils in a larger way than to prove certain spatial facts? Some of the answers are here attempted.

1. Organization around broad concepts. The second report of the Commission on Post-War Plans states: "The work of each year should be organized into a few large units built around key concepts and fundamental principles." As it concerns demonstrative geometry, the "key concepts and fundamental principles" may not be exclusively spatial concepts like quadrilaterals or similar triangles. It is obvious that learning to think about a trapezoid helps one not at all to think about communism or democracy.

The key concepts should be the nature of proof, the assumption, the definition, the theorem, the general proposition, and other such ideas. The average teacher is probably thinking that he has been doing this right along, that is, he has explained these ideas to his class. But this is not quite the same as saying to a class: "Today we shall begin the study of the assumption which will occupy us for about three weeks. We shall learn why we cannot prove everything and that we usually have a choice in assuming some things and proving others from them. Some of these assumptions are in the form of definitions. We shall learn what a definition is, so that we will recognize it whether we meet it in geometry or elsewhere." It may be asked why it is necessary to spend three weeks on this idea. When we consider the countless numbers of pupils who have completed courses in demonstrative geometry without assimilating the fundamental fact that the propositions in geometry depend on assumptions, the answer becomes evident. In fact, the idea will have to be kept before the pupil throughout the course.

The function of the theorem in a system of consecutive thinking is another general concept that must be developed patiently and deliberately. Paradoxically, it is possible to teach many theorems without teaching *the* theorem. How often do pupils cite the theorem they are trying to prove as an authority in the proof of the same theorem? How many pupils, when told to apply a certain theorem to an original problem, will start from scratch and repeat the proof of the theorem without using the theorem itself? The term *corollary* is an everyday word. Do pupils understand its relation to a theorem or do they sometimes confuse it with a postulate? Do they appreciate that the proof of a theorem, in addition to being logically correct, should also have a certain dramatic effect which is achieved by bringing into the proof the facts only as they are needed to bolster the argument? Some pupils bring down all the given data at the outset of the proof and may thus be compared with the scenario writer who has his hero stop the train ten miles from the burning trestle. It is clear that the theorem, which may be likened to the formula in algebra, is a generalized concept worthy of study apart from its factual content.

Another generalized concept is the idea of *determinism* so that the following notions are unified: A set of conditions in the hypothesis of a proposition determines a necessary (logical) conclusion; two points determine a straight line; two quantities determine a sum, a difference, a product, or a quotient; two angles of a triangle determine the third angle, etc.

In some texts, the material is now organized along such generalized concepts. Surprisingly little adjustment in the traditional sequence is necessary to secure this end. Many have tinkered with geometric sequences to achieve a more teachable geometry. However, few changes are really needed to play up the large divisions of generalized ideas as the essentials of clear thinking which are inherent in Euclidean geometry. This encouraging fact gives much promise for the future of

geometry—a future that can establish geometry as the backbone of straight thinking in the secondary-school program.

2. Broad terminology. To make the thinking general enough to apply to other than spatial situations, our terms must be made sufficiently broad at the *outset*. If we restrict a *proposition* to a theorem or a construction, we must later utilize the controversial principle of mental transfer to consider non-mathematical propositions. Wider significance is developed if a proposition in geometry is likened to an equation in algebra, which in turn is compared to a sentence in English. The outmoded distinction between axioms and postulates has no place in generalized thinking which consists of two kinds of propositions—proved and unproved. A definition clearly belongs in the unproved category. Surely, a definition is different from an ordinary "if-then" proposition. The difference consists of the fact that, by virtue of its reversibility, a definition contains two propositions of which each is the converse of the other but both are assumed. For those who still have reservations about this, we may cite the fact that a set of assumptions is often used to characterize and thus define a situation. As an example, consider the proposition that if a line is drawn through any two points in a plane, the line is entirely in the plane. This is sometimes regarded as a postulate and sometimes as a definition of a plane. In either form, the proposition defines a plane.

The idea of necessary and sufficient conditions is another concept that is useful in expressing relationships in mathematics. Yet, for some reason, this concept has not been used in elementary mathematics even though *necessary* and *sufficient* are common everyday words. If we really seek a generalized mode of thinking we may well recognize the effectiveness of necessary and sufficient conditions to express all-inclusive relationships both in mathematics and elsewhere.

Analogy is still another term that has potentialities in a broad structure of thinking. Many fallacies in non-mathemetical thinking are the result of false or imperfect analogies. In the concept of similar polygons we have the perfect formula for thinking by analogy. Here is a wonderful opportunity to develop the pattern of analogy at the outset and cite similar polygons as the ideal with which other so-called analogous situations may be compared and evaluated.

3. Stress on methods. We must patiently examine and compare various methods used to reach conclusions. Great emphasis should be laid on these methods as *modi operandi* for reasoning in a variety of situations. The stress must be not only on reaching the conclusions in individual propositions, but must also include an over-all picture of the sequence of propositions. This is best accomplished by picturesque dependence charts which the pupil can grasp at a glance without having to read fine print or refer to propositions in scattered places. The charts should be accompanied by pointed questions which will oblige the pupil to seek out meaningful relationships.

The indirect method must not be used as a mouse-trap device for forcing the pupil into admitting a foregone conclusion. Instead, it should inevitably be bound up with the nature of a proposition, its converse, its inverse, and so on. Considering the extensiveness of this method in everyday thinking, it would seem that the method has been practically ignored in demonstrative geometry. Certainly, the picayune non-mathematical applications which are usually cited do not raise the prestige of geometry in the eyes of the pupils. Among the methods to be stressed are various problem-solving techniques such as trial-and-error and analysis. To implement these techniques the pupil should learn the principles of classification and its uses.

4. Induction and deduction. The recent trend has been to introduce more and more of the inductive method. However, the motive behind this may be questioned.

It appears that this is being done to escape "formal proof" which is considered too difficult, especially for the beginner. How the pupil is being prepared for deductive proof by doing something else is not quite clear. In this connection, it is interesting to ask why the pupil should be taught the Euclidean constructions by rote and limited to them in his experimental work. If deductive proof is really so difficult, the solution would appear to be a gradual approach by way of two- or three-step simple proofs. For the study of geometry to encompass a general way of thinking, induction must of course be an integral part of the work. But it must not be an escape from deductive proof, nor must it be a deprecation of the inductive method. The inductive and the deductive methods must be taught side by side because an understanding of each is necessary to a proper understanding of the other. What is needed here is a form of straddling to develop two parallel ideas.

The following may be cited as another example of straddling in an effort to generate broad concepts. In the early stages of the work the pupil is asked to comment on the validity of the proposition that if a person is red-headed then he is ill-tempered. The pupils will condemn this as not agreeing with observed facts. This, of course, is not the point of the exercise and so the class is asked to seek the evidence that is needed to make the proposition valid. Some pupils will quickly discover the principle of cause and effect; i.e., evidence is needed to show that the pigment of the hair causes ill-temper. This gives the pupil an approximate idea of the difference between inductive and deductive proof. Immediately following this exercise, the pupil is given a diagram of two intersecting lines and asked to discover two equal angles. At this stage the pupil may not even have learned the meaning of vertical angles but he has learned something about the size of an angle in terms of rotation. All pupils will quickly find a pair of equal angles. When asked why the angles are equal, they may be expected to say, "we can see it" or "we can measure them." Now the pupils are told that they are giving experimental evidence but, as in the case of the red-heads, we seek cause and effect. Without further prompting, some pupils will say that both angles are the result of the same amount of rotation. This again illustrates the difference between inductive and deductive proof.

With the aim of giving real meaning to the inductive and deductive types of proof, the early sequence of propositions may be such that parallel lines are studied before congruent triangles. The only angle relations that the pupil studies at first are those that result from the intersection of lines, including parallel lines and transversals. At this point, therefore, the pupil has not seen any logical relationship between the equality of line segments and the equality of angles. He is thus confronted with a real problem for the solution of which he sees a definite need and for which he is taught to use the inductive method.

In evaluating the soundness of certain conclusions, the question of how these conclusions were obtained is often of considerable importance. For this reason, a well known exercise in deductive thinking is to have the pupil identify the underlying premises. This device should be extended to statements where the pupil is asked to determine whether inductive or deductive thinking is used to reach a given conclusion.

5. Generalization and illustration. The ability to illustrate a situation from a general statement and the reverse, that of generalizing a particular situation, is usually assumed to be acquired in an incidental manner. The fact that it does not work out that way is not the only reason for making a separate study of it. A more important reason is that this ability has wide application outside mathematics. In a study of this kind, the relation between general and special cases can be

made to generate more insight into geometric figures. Thus, the area of a parallelogram and that of a triangle can be made special cases of the area of a trapezoid. Two triangles that agree in two of their angles form a special case of two triangles that agree in but one angle. From the fact that the areas of the latter triangles are to each other as the products of the sides forming the equal angles, it is easily demonstrated by straight-away algebra that triangles agreeing in two angles have their sides proportional.

For the faster pupil, Ceva's theorem can be brought down to the high-school level as follows: The concurrent lines in Ceva's theorem divide the original triangle into six triangles of which three pairs have common altitudes and three pairs have equal vertical angles. The six resulting proportions can be multiplied so as to yield Ceva's theorem. This theorem, in turn, can be seen as a generalization of the altitudes, the medians, the angle bisectors, and other concurrent lines in the triangle.

The study of the angular measurement of arcs may be begun with the obvious fact that the sum of two vertical angles at the center of a circle equals the sum of the intercepted arcs. This can be generalized by moving the vertical angles away from the center, the sum of the arcs becoming an algebraic sum as the vertex of the angles moves outside the circle. The inscribed angle and other angles that intercept arcs on the circle then become special cases.

6. Using motion and change. Because rigorous proof in geometry is concerned with stationary figures, the idea of motion and change of position has received but scant attention. The study of figures in motion is a powerful tool for building generalized concepts. The relation between vertical angles through the idea of rotation and the study of the angular measurement of circular arcs by moving (translating) vertical angles over a circle have been mentioned above. The use of motion is one important way to teach Gestalt patterns that show at a glance various special cases. The study of a line as it moves over a circle shows at a glance the important relation between a secant and a tangent, enables the pupil to learn propositions about both simultaneously, and prepares him for the further study of mathematics where the tangent is defined as the limiting position of the secant. It gives the pupil two conceptions of a locus: (a) the static idea of a figure consisting of all points of a certain type, and (b) the dynamic path of a moving point.

The ability to study a changing situation and abstract from it a constant relationship is most valuable in mathematical, in scientific, and in general thinking. The invariant property of a changing triangle which retains its angles is the basis of trigonometric functions. The invariant in a changing circle leads to an understanding of the constant π. The invariant in an expanding gas is the law of gases. The invariant in our changing social order is the basis of our democratic society—a dynamic concept that is so vital in secondary education.

7. Deformalization. To achieve the flexibility that is needed to establish geometry as a mode of thinking, the rigidity that still attaches to the subject in various forms must be relaxed. The principle that *all* the authorities required in a demonstration must be stated explicitly is called into question. It is responsible for the formulation of assumptions like, "A point is within, on, or outside a circle according as its distance from the center is, etc." This artificial assumption is used only once to make a certain proof "rigorous." In effect it does nothing more than encourage rote learning. It is much better to teach proofs which do not call for such improvised assumptions. Where this is not feasible, the pupil should be allowed to use an authority which has general application even if it applies only indirectly to the given situation.

In general, we must come to accept the imperfect. Within bounds, it is more in-

structive to accept a pupil's version of a proposition, a definition, or an authority than to insist on bookish statements. We all know the practical results of the latter. It is more important for the pupil to know the source of his authority, such as theorems or definitions, than to reproduce a preconceived form. The aim of developing precise expression cited earlier does not entail uniform perfection. Let us note that the pupil's ability in this respect is a changing quantity. Our main concern is that it be a growing quantity. The conventional form of the parallel postulate is a case in point. The pupil sees intuitively that parallel lines have the same direction The magnetic compass shows him that the direction of a line is the angle which the line makes with a fixed reference line. He is thus led to see that lines which make equal corresponding angles with the refence line are parallel. To have a workable parallel postulate that agrees with everyday experience, it is only necessary to assume that any transversal (not only the north-south line) may be used to measure directions.

The conventional treatment of the equality of arcs is another instance that leaves room for deformalization with its accompanying increased meaning. Although the method of superposition to prove triangles congruent has been generally replaced by postulation, it still persists in the proof of equal arcs. It is more significant to teach that an arc has two measures, angular degrees and length. The degree measure of an arc is the only measure needed to study the angles and chords in a circle. This measure can be defined simply as the number of degrees in the central angle. The length of an arc is not needed until the circumference of the circle is studied near the end of the course. There the lengths of the arcs in the same circle with equal central angles may be taken up as a corollary to the simplified version of limits that is usually studied.

Deformalization leads to simplification. Simplification of traditional geometry is needed to make the subject more meaningful, to save time for developing generalized concepts, and to attract more pupils of moderate ability. Some of the possible simplifications are implicit in the remarks made so far. A separate diagram for each exercise with data marked on it makes it easier for the pupil to concentrate and saves time in classwork. Mechanical proofs can be replaced by algebraic exercises. It should be recalled that many conventional proofs date back to the time before algebra was developed. For the modern school to ignore the algebraic method is like tying one hand behind the back. Besides, there is no better way to maintain skill in algebra. Another obvious way to simplify the work in geometry is to avoid repetitions. Thus, the theorem about parallels intercepting equal parts on a transversal can be omitted because it is a corollary to the more general proposition about parallels intercepting proportional segments on several transversals.

Conclusion. The program for making the general values of geometry independent of the theory of mental transfer has thus been indicated. Perhaps these suggestions appear like a major reformation and some may ask whether it is necessary or worthwhile. To answer this let us look briefly at the present status of demonstrative geometry.

Geometry is usually studied by pupils because of college requirements. For every pupil who breezes through the subject and enjoys solving the originals there are many more who struggle to keep up with an artificial set-up.

We have made extravagant claims for the subject as an aid to thinking. Judging by the dwindling numbers in our classes, we have not convinced the curriculum makers of these claims. Men and women who studied the subject in school are not convinced and often ask why they had to take it. Even some who went on to take college mathematics never understood quite clearly the function of demonstrative geometry aside from teaching them certain

spatial facts which they could have learned more quickly by intuition.

The most serious indictment against us is that we have not been true to our claims. Smug in the belief that the broadening values of geometry function automatically, we have been interposing obstacles to suggested changes. We have often cited the immaturity of the pupils to study generalized concepts. If this is a valid objection it, of course, nullifies our important claims for geometry. Now, that achievement tests no longer stress formalism and are more concerned with insight, the contention that geometry has a fixed scope has lost much of its force. Notwithstanding the fact that many pupils have found geometry so difficult that they had to resort to memorizing, we still prefer to cover a predetermined amount of this work rather than teach a little of the more enduring values. The lack of suitable material to teach geometry effectively has been partly remedied. Pioneers like Fawcett, Christofferson, and others have pointed the way and have suggested interesting and meaningful methods for generalizing. It remains for the serious teacher to try out this material in class. By condemning all the plans that have been suggested without a class trial, we may be forced to surrender our claims to the real values of geometry. This would lessen the prestige of the subject and would probably be reflected in the college requirements.

There are scattered groups of teachers who are making serious attempts to realize the broader purposes of geometry. The teachers of Newark, New Jersey, for example, have adopted a course of study with that general aim. These teachers have found the new work no more difficult than the conventional material. The fact is that many of the pupils who would normally experience difficulty with the standard work on spatial relations, find the broader ideas easier to understand because they come closer to things with which they are familiar. These teachers find increased satisfaction in the knowledge that their pupils learn ideas that they will appreciate more and more as they meet and use them in other fields. They are proving the worth of demonstrative geometry no matter what the ultimate fate of mental transfer.

Don't You Think . . . ?

We ought to give prizes
Of various sizes
For humor that capers
On Algebra papers.

To Shirley a rose and a volume of Plato's:
She carefully labelled her "asympotatoes."

To Glenmore a scarf and a special salute
For discarding an "extemporaneous root."

KATHARINE O'BRIEN
Deering High School
Portland, Maine

We Took the Class Census

By Fay Layne

Laboratory School, University of Chicago, Chicago, Illinois

The 7's* were busily discussing the interpretation of per cents. What per cent of the class had blue eyes? What per cent was 12 years old? The nineteen pupils and the teacher made a group of twenty which was easily interpreted on the basis of 100.

The discussion continued with the interpretation of percentage clippings each pupil had collected. Some of the statements concerned populations which naturally turned attention to the then current census.

"Why don't we take a seventh grade census?" mused one of the pupils.

"That's an idea."

"But it would take a lot of time."

"Do you think we could do it?" asked one pupil turning to the teacher.

The boys and girls were accustomed to sharing in the planning of their work. They knew we were just starting a unit for the purpose of gaining more experience in the use of per cents.

"The answers to census questions could be worked out in per cents," one pupil pointed out.

"Then we could make graphs."

"Circle graphs would give us a chance to use our protractors again."

"We could put the graphs in a booklet."

"There should be an editor for the booklet."

"We'll need census takers. I'd like to be a census taker."

"I don't know, but I think the homeroom teachers would let us take the census in homerooms, so we could ask everyone in the grade."

The opinion was expressed that although a number of the pupils were enthusiastic about the idea, it would not be worthwhile to give such a project any consideration, unless the group as a whole was interested in it. Every child indicated that he would be willing to work on the project. They also realized that it would not be fair to ask their classmates to cooperate by answering census questions unless they fulfilled their responsibility of sharing the results with the total grade. The more they thought about the project, the more they felt they could succeed.

A chairman and a secretary who was to keep minutes of the plans made by the group were elected. Each pupil, as well as the teacher who was seated among the children, was to be recognized by the chairman before addressing the group when any business pertaining to the census was being conducted. After the elections the chairman asked each member of the class to prepare a list of questions that he thought would be suitable for the seventh grade census and to bring them to class the next day.

At the next class session the group decided that no more than ten questions should be used in the census and that the basis for selecting each question should be its interest to seventh graders. After much discussion as to which questions to include and as to how they should be worded, the following list of questions was finally selected:

Census Questions

1. What is your favorite sport?
2. Do you live in an apartment or a house?

* The total seventh grade was divided into three classes approximately equal in size. The children were not grouped by ability. The numbers 1, 2, and 3 were merely used to designate sections. Each class in mathematics met for a 50-minute period four days per week. The whole grade was also divided into five homeroom groups. Each homeroom met for a 10-minute period each morning before classes started and for a 50-minute afternoon period once each week.

3. What are your hobbies?
4. How many brothers and sisters do you have?
5. What pets do you have?
6. How much is your allowance?
7. What is your height and age?
8. Where were you born?
9. What foreign countries have you visited?

As the project began to take form the class was divided into three committees. There were 5 census takers, 8 tabulators, and 6 pupils serving on the booklet committee. Each committee elected its own chairman. The teacher moved from group to group and sat in with the different committees as they asked for help and needed suggestions in getting organized.

The work of the census takers involved interviewing all seventh grade pupils. They worked out a form for recording their data. The form was made to correspond with the list of census questions, so that the data collected from each pupil could be written in a column during the interview. The names of the children questioned were not recorded in deference to the question about allowance.

The five census takers obtained permission from their respective homeroom teachers to interview pupils during the homeroom period. After the first period of interviews the enumerators reported that some of the boys and girls who would soon have birthdays were uncertain about the number they should give for their ages. After the class gave this problem some consideration it was agreed that the age question should be changed to read, "How old will you be on June 14?" The ages would then be correct by the time the booklet was prepared. The children who had already been questioned were to be checked again to keep the record accurate in regard to ages. The census takers also reported that all of the rest of the questions seemed satisfactory.

The members of the tabulating committee organized the information provided by the census takers in preparation for designing the graphs. They started their preliminary work by discussing methods for dealing with each question and each pupil took the responsibility for one or more questions. The first step was to arrange the responses into frequency tables. A number of categories on the tables contained single items. The single items were then regrouped into categories labeled "miscellaneous." The committee was given the freedom to use its judgment in grouping data other than single items. After the data were classified the committee worked out per cents, each rounded off to the nearest whole per cent, based on the 55 members in the grade.

Members of the booklet committee took the data given to them by the tabulating committee and made suitable graphs. They also worked out the general plan of the booklet which included the cover design and the plan for the second page which gave recognition to the personnel of the various committees. A rough draft of each proposed graph was sketched and then, after checking and correction, was drawn on a master carbon ready for duplication. Another phase of their work was the duplication of enough copies from the master carbons to prepare a copy of the complete booklet for each boy and girl in the seventh grade.

The total class time was not devoted to the census project during the interval between the initiation of the unit in April and the completion of the booklet on June 14, 1950. As a class they reviewed decimals, made some drawings to show the relationship between equivalents, rounded off per cents to various decimal places, worked with some other applications of per cents, saw a film on percentage, did some work with the fundamental operations which was a part of their regular maintenance program, made a general study of different kinds of graphs and how to draw them, and spent several periods on tests. In addition to the whole-group activities, there were always puz-

zles and references to supplementary topics available for individual pupils who had the time and interest to pursue them.

As the project progressed more of the time was devoted to small group and individual work. The boys and girls from any one of the three committees might be working together in groups of one, two or three as the booklet approached completion. A pupil who had done the tabulating on a certain question would often collaborate with a pupil from the booklet committee who was making the final graph. Pupils from the census taking committee helped booklet committee pupils in planning the scales and in selecting the proper kinds of graphs to illustrate some of the questions. Most of the pupils preferred to help with an unfinished piece of work rather than do some non-census activity of their own choice, if their original assignment on the census project had been completed. At this stage the first few minutes of each class period were used to see that each individual knew what he could do. The membership of the small groups became very fluid as some jobs were finished. After consulting with the chairman and the teacher the unemployed pupils formed new groups starting on new tasks or affiliated themselves with groups already at work where more help was needed. Through this procedure every student actively participated in the actual process of the interpretation of the data into graphs.

The teacher advised many of the small groups as they needed help with the per cents and graphs. The pupils were asked to keep all of the sheets on which the enumerators had made recordings and all of their tabulating and figuring. Although the students had had some experience with making graphs, the selection of the kind of graph that would best represent their data, the selection of a suitable scale, planning the size of a graph to fit one-half page of standard typing paper, and the use of the privilege of using originality to make a graph interesting without sacrificing accuracy and clearness did raise some occasions where students needed counsel.

The responses in answer to the question, "What are your hobbies?", produced the data the group found most difficult to interpret to their satisfaction. There were 81 responses with 35 different hobbies listed only once. The frequency table for these data was:

Hobby	*No. of Children*
Miscellaneous	35
Stamps	19
Post cards	6
No hobby	6
Books	3
Dogs	3
Fishing	3
Piano	3
Swimming	3

The 7[1]'s felt accountable for making graphs from which their classmates could identify and interpret the answers they had given to the census questions. Lumping only a few single items into a group marked "miscellaneous" would have been readily accepted by the group. But they were disappointed to see roughly half of their interesting list of hobbies lost in the obscurity of a miscellaneous category.

They could see that per cents based on the 55 members of the grade would total more than 100% and would not fit into a circle graph. They could also see that a circle graph made from per cents based on the 81 responses would be impractical because the last six categories on the frequency table would be represented by very small sections.

They considered making a bar graph by retaining the 35 specific hobbies mentioned once. While still in the visionary stage they could see that this type of graph would have too many bars to fit on one page. The members of the class were also dissatisfied with all of their attempts to arrange the graph on two consecutive pages.

Their final decision was to make a bar graph using "number of pupils" along the scale. It was anticipated that the

bar representing the miscellaneous items would be large enough that the separate items could be listed in paragraph form on it or that a list could be placed on some other part of the graph. On the rough draft of the graph there proved to be too many items to list all of them on one bar. When the last draft was made on the master carbon it was decided that the graph would look too cluttered with an explanatory itemized list above the shorter bars. After much experimentation the final graph was a vertical bar graph whose longest bar bore the single word, "miscellaneous."

Although the line graph was not used to interpret any of their data it was considered in some of their group discussions. They eliminated the use of the line graph after comparing their data with the types of data which line graphs were used to represent. They were aware that more than one kind of graph could have been correctly used in some cases but made their choices intending to give over-all variety to their complete collection. Several less experienced pupils found that numbering a scale beginning with zero and numbering the lines made a graph easier to read than their first attempt to number spaces along the scale. There was no instance of any student failing to recognize the difference between "number of pupils" and "per cent of pupils" in labeling the scale of a graph he had made.

In addition to the graph about hobbies which has already been discussed there were nine other graphs in the completed series. A summary of notes about the other graphs appears below. The data collected by the students, after arrangement into categories, are given in tabular form beneath each question. The column on the left contains the responses given in answer to the census question. The column to the right lists the number of pupils giving each response. Following each table is a brief statement describing the kind of graph made to interpret the foregoing data.

1. *What is your favorite sport?*

Baseball	34
Swimming	6
Basketball	5
Miscellaneous	5
Riding	3
Ice hockey	2

Kind of graph: Circle graph with each section labeled with the nearest whole per cent.

2. *How much is your allowance?*

Under $.75	14
$.75 to $1.50 inclusive	24
Over $1.50	17

Kind of graph: Picture graph on which a stick man represented each student in the grade.

3. *Do you live in an apartment or a house?*

House	22
Apartment	33

Kind of graph: Circle graph labeled with the nearest whole per cents. Pictures of a house and an apartment building illustrated their respective sections.

4. *What pets do you have?*

Dog	24
Fish	19
Cat	7
Bird	3
Turtle	3
Horse	2
Rabbit	2
Miscellaneous	5
None	22

Kind of graph: Horizontal bar graph with a scale labeled "number of pupils."

5. *How tall are you?*

5′4″ and over	10
5′0″ to 5′4″	29
Under 5′0″	16

Kind of graph: Vertical bar graph with "per cent of pupils" along the vertical scale.

6. *How many brothers and sisters do you have?*

One	25
Two	15
Three	4
Six	1
None	10

Kind of graph: Circle graph labeled with per cents.

7. *How old are you?*

11 years	1
12 years	37
13 years	15
14 years	2

Kind of graph: Picture graph on which each symbol of a child represented 10% of the seventh grade.

8. *Where were you born?*

Chicago	39
Boston	2

New York	2
Other places	12

Kind of graph: Circle graph based on per cents. The 12 miscellaneous places including two foreign countries were listed below the graph.

9. *What foreign countries have you visited?*

Canada	36
Mexico	7
France	3
Spain	2
Italy	2
Other countries	13
None	27

Kind of graph: Horizontal bar graph with "per cent of pupils" along the scale.

The Commission on Post-War Plans in reference to the mathematics program for grades 7 and 8 states in part: "In building for power the classroom work should involve more than mere theory. It calls for constant and insistent attention to significant applications. This is necessary for the sake both of effective motivation and of effective transfer."* When an activity is important to boys and girls on its own merits and they have pride about its completion, it possesses some factors which meet the true test of significance for those students.

Although the data described here contain relatively small numbers for statistical work, the motivation and understanding such material provides justifies its use as part of the program for teaching basic concepts. The concepts toward which a contribution was made were not limited exclusively to those associated with the finished graphs. In many cases the pupils considered a range of possible interpretations and had the opportunity of exercising discrimination in their decisions. Some of the students were inspired by the responsibility of being the sole source of some unknown kinds of information about their grade.

The reader may have gained the impression that the program in mathematics is built on the momentary whim of random pupil interest. In practice this is not the case. Many parts of the program are known to be necessary and are accepted as such by the students. Since the pupils frequently discuss what they need to accomplish in mathematics, they often display considerable insight in helping to set up the day-to-day plan for carrying out the details of the general aims of a new unit and can see the purpose and value of selecting certain class activities. While the teacher has over-all plans for what is to be accomplished, the program is flexible enough to allow for variation in method from class to class in developing a particular unit.

Because junior high school pupils are not mature enough to work together with "Utopian" skill in small groups, this method does have limitations and would be only one of the kinds of experiences used in mathematics. Students who may be dormant or dominant members in whole-group activities are less frustrating to their classmates because of the teacher's guidance in that situation. However, the lack of tact and skill which comes to light in some pupils may be offset by the display of previously unused talent and insight other students demonstrate in this kind of classroom atmosphere. Mathematics teachers are accustomed to observing students at work in order to diagnose their difficulties in performing the fundamental operations. They, along with teachers in other fields, are on less familiar ground in sharing the more complex responsibility of observing boys and girls at work in groups and in helping them to improve their ability to work co-operatively with others.

In adapting the type of unit described here to the use of smaller or larger classes, the delegation of tasks to individuals or to both individuals and small groups may be advantageous in accommodating the number of children involved. Even though the items on the list of census questions given here would not have equal significance for other boys and girls, most of them are suggestive of what is interesting to junior high school pupils.

* "The Second Report of the Commission on Post-War Plans," THE MATHEMATICS TEACHER, XXXVIII (May, 1945), 205.

Pisa, Galileo, Rome

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In Professor W. R. Carnahan's article, "Mathematics an Essential of Culture," which appeared in the November, 1950 issue of this magazine, there occur certain oft-repeated statements which, in the light of recent research, appear in need of revision. These statements concern Galileo's experiments with falling bodies from atop the Tower of Pisa and Galileo's clash with ecclesiastical authorities. As a matter of fact the reiterated statements concerning Galileo and Aristotle, and Galileo and the Catholic Church tend to obscure Galileo's real merits as a scientist. Even in Professor Carnahan's article we do not have as detailed a discussion of Galileo's contributions to the science of mechanics, on which his right to a place in the history of science properly rests, as we should have expected.

Concerning "the famous gravity demonstration at the Tower of Pisa" the writer would like to recommend an admirable little book by Professor Cooper.[1] There one learns that the story of these experiments was first written by Viviani, Galileo's earliest biographer, about twelve years after Galileo's death and more than sixty years after the assumed date (*c.* 1590) of the experiments. Since Galileo meticulously committed to writing and published[2] the results of his physical experiments and deductions it is strange that he should have omitted any mention of these "famous" experiments. It is true that in Galileo's writings we find the material on which, in all probability, Viviani built his story. In criticizing Aristotle's ideas on motion Galileo asks the following question in *De Motu:*

> "... if two stones were flung at the same moment from a high tower, one stone twice the size of the other, who would believe that when the smaller was half-way down the larger had already reached the ground?"[3]

Again in Galileo's last and most important work, *Discorsi e Dimostrazioni Mathematiche intorno a Due Nuove Scienze,* we are told in the words of Salviati, one of the characters carrying on the dialogue:

> "Aristotle says 'An iron ball of one hundred pounds, falling from a height of one hundred cubits, reaches the ground before a one-pound ball has fallen a single cubit.' I say that they arrive at the same time. You find on making the experiment, that the larger precedes the smaller by two finger-breadths; that is, when the larger one has struck the ground, the other is short of it by two fingers. Now you would not conceal behind these two fingers the ninety-nine cubits of Aristotle."[4]

Nowhere does Galileo state that he himself carried out such experiments. But we know definitely that others did.[5]

However, more important than the actual performance of such experiments is the defamation which the story throws on Aristotle as a physicist. Nowhere in Aristotle's works can be found the statement attributed to him by Galileo through the mouth of Salviati and which has been continuously associated with the various ramifications of Viviani's story. Arguments against Aristotle are usually based

[1] Lane Cooper, *Aristotle, Galileo and The Tower of Pisa,* (Cornell University Press, 1935).

[2] Among the few exceptions is *De Motu,* written about 1590 but not published until 1883.

[3] Galileo Galilei, *Le Opere di Galileo Galilei,* (ed. A. Favaro), (Florence: 1890–1907). Reprint 1929, Vol. 1, p. 263. Trans. by Cooper, *op. cit.,* p. 83.

[4] *Ibid.,* Vol. 8, p. 109. Trans. from *Dialogues Concerning Two New Sciences* by H. Crew and A. De Salvio (New York: The Macmillan Co., 1914), pp. 64–65. Hereafter referred to as *Two New Sciences.*

[5] See Cooper, *op. cit.,* for details and also interesting examples of variations of the story. See also, H. Butterfield, *The Origins of Modern Science,* (New York: The Macmillan Co., 1951), p. 59 ff.; H. T. Pledge, *Science Since 1500,* (London: Ministry of Education, 3rd reprint, 1947), p. 61.

on quotations from his *Physica* and *De Caelo* where he proposes proofs for the nonexistence of a void or vacuum. But as Hardcastle pointed out:

"What he [Aristotle] taught was that the terminal velocity of a heavy body was greater than the terminal velocity of a light body." . . . "Aristotle throughout treats only of the motion of projectiles, and of that only in a resisting medium, and then only of that part of the vertical motion when the projectile has attained that constant speed known to ballisticians as 'terminal velocity' which can be as readily observed in rising smoke as in falling rain."[6]

One from many possible quotations from Aristotle should suffice to substantiate this interpretation:

"We see the same weight or body moving faster than another for two reasons, either because there is a difference in what it moves through, as between water, air and earth, or because, other things being equal, the moving body differs from the other owing to excess of weight or of lightness. Now the medium causes a difference because it impedes the moving thing, most of all if it is moving in the opposite direction, but in a secondary degree even if it is at rest, and especially a medium that is not easily divided, i.e., a medium that is somewhat dense."[7]

It is true there are passages in Aristotle's physics which seem to offer indirect evidence in proof of Galileo's viewpoint. But it must be remembered that, first and foremost, Aristotle was a realist, and not a dreamer, as he has been sometimes portrayed. From ordinary experience it should have been obvious to him that similar bodies of different weight would fall through a given distance in practically the same time. It is difficult to imagine that the man whose physical theories have recently been so highly estimated by Professor Boyer[8] could have decided otherwise and with such a discrepancy from reality. It may be pointed out here that Aristotle's meaning in many passages has been obscured. Through errors in the copying of early manuscripts and the difficulty of determining the actual meaning of many of the technical terms employed a variety of translations is possible, as may be seen by even a comparison of the translations given here with those in The Loeb Classical Library editions.

In any examination of Aristotle's physical theories it is important to bear in mind, as Professor Boyer points out, that to the Greeks "*physis* meaning the essence or nature of things, was more concerned with the explanation of essential properties and relations than with quantitative description." Professor Boyer's estimate of Aristotle's scientific achievements is of importance in the light of later developments in the physical sciences:

"The physical science of Aristotle is a coherent and systematic treatment which, while less accurate than that of Archimedes, is of far wider scope. In method there is a striking similarity between these two great scientists of antiquity. Both men began with careful observation unaided by the use of instruments, framed inductive generalizations and with these as premises built a deductive science. But while Archimedes, a mathematician, limited himself to a few observations comparable in exactitude to the axioms of Euclid's *Elements* of geometry, the philosopher Aristotle surveyed the whole of nature so that he might disclose the rational order in cause and purpose. For this reason Aristotle placed the inner consistency of his system above accuracy in detail, and in his eager search for certainty he failed to exercise the needed suspension of judgment."[9]

Criticisms of Aristotle's statements on falling bodies and other sections of his physics did not begin with Galileo, they date back at least as far as the period of Philoponus in the sixth century. The modern concept of the content of physics as a science which seeks proximate causes in contrast to Aristotle's concept which embraced ultimate as well as immediate

[6] J. H. Hardcastle, Correspondence Section, *Nature*, (June 25, 1914), p. 429. See also (January 22, 1914), pp. 584–85.

[7] Aristotle, *Physica*, 4, c. 8, 215[a] 25–31. Trans. by Hardie and Gaye, *The Works of Aristotle*, (ed. W. D. Ross), Oxford University Press, 1930.

[8] Carl B. Boyer, "Aristotle's Physics," *The Scientific American* (May, 1950), pp. 48–51. See also by the same author "Quantitative Science without Measurement: The Physics of Aristotle and Archimedes," *The Scientific Monthly*, (May, 1945), pp. 358–64.

[9] *Ibid.*, p. 50.

causes came prominently to the fore during the thirteenth century. Two important innovations of this century were the increased emphasis on the mathematical or quantitative aspects of the subject matter of physics and the systematic development of the experimental method for the verification of scientific hypotheses. The earliest important depositions on these fundamental changes are to be found in the writings of Bishop Grosseteste of Lincoln,[10] for many years prominently connected with Oxford University. But it was in connection with the quantitative study of motion that the most fundamental rupture occurred between the advocates of the new physics and the adherents of Aristotelian physics. Through the critical attacks on Aristotelian ideas on motion, carried on particularly at the Universities of Oxford and Paris by such men as Occam, Buridan, Duns Scotus, Suiseth, Bradwardine, Roger Bacon, Albert of Saxony and Bishop Nicholaus of Oresme, there arose the mathematical theory of accelerated motion which reached its final form with the development of the calculus by Newton and Leibnitz, and also the theory of impetus which Descartes was later to develop into the theory of inertia.[11] In the meantime, during the fifteenth and sixteenth centuries, the study of natural phenomena by the so-called adherents of Aristotelian-Scholasticism had sunk to a low ebb and as a result the whole science of Aristotle was repudiated. As Whittaker, the well-known British mathematical-physicist, has expressed it:

"From the fourteenth century onwards, Scholasticism was decadent and by the end of the sixteenth it had become thoroughly debased. The love of nature that had been so vital in Aristotle had almost perished; the practice of observation and experiment, on which he and St. Thomas had so strongly insisted, was neglected save by a few solitary workers; and the degenerate Schoolmen occupied themselves with futile subtleties that bore no relation to life and reality."[12]

Or again quoting Professor Boyer:

"By the time of Galileo and Newton the science of Aristotle had been thoroughly discredited. It became the fashion to ridicule Aristotelian science for its errors, more putative than real."[13]

The fashion still seems to persist, but in view of the high esteem in which the work of Aristotle is held by many modern historians of the physical sciences and mathematics future critics of Aristotelian physics should be deterred from endeavoring to bury "The Philosopher" beneath hundred-pound balls dropped from atop the Tower of Pisa. All the more so when it is kept in mind that the greatest derider of Aristotle in the seventeenth century was himself a rather haphazard physicist even by medieval standards. It is rather difficult to overlook Galileo's disregard of experimental verification as exemplified in the following:

"Galileo worked out his physics by thought, by correct reasoning and mathematics, not by induction from experiments. During his days at Pisa, before he went to Padua, he wrote: 'But, as ever, we employ reason more than examples (for we seek the causes of effects, and these are not revealed by experiment).' Galileo liked to use what he called 'Thought Experiments,' imagining the consequences rather than observing them directly. Indeed, when he described the motion of a ball dropped from the mast of a moving ship, in his *Dialogue on the Two Great Systems of the World*, he then had the Aristotelian, Simplicio, ask whether he had made an experiment, to which Galileo replied: 'No, and I do not need it, as without any experience I can affirm that it is so, because it cannot be otherwise.' "[14]

[10] G. Sarton, *Introduction to the History of Science* (Baltimore: The Williams and Wilkins Co.), Vol. 2, p. 583 ff.

[11] P. Duhem, *Études sur Léonardo de Vinci*, Vol. 3, "Les Précurseurs de Galiléi," (Paris, 1913). A. Koyré, *Études Galiléennes*, (Paris, 1939). Carl B. Boyer, *The Concepts of the Calculus*, (New York: Hafner Co., 1949), p. 71 ff. H. Butterfield, *op. cit.*, Ch. 1.

[12] E. Whittaker, "Aristotle, Newton, Einstein," *Science* (Sept. 17, 1943), p. 250.

[13] Carl B. Boyer, *op. cit.* (8), p. 50.

[14] I. B. Cohen, "Galileo," *Scientific American* (August, 1949), p. 45. For further elaboration on Galileo's "Thought Experiments" see, E. A. Burtt, *The Metaphysical Foundations of Modern Physical Science*, London, 1949, p. 65 ff. and H. Butterfield, *op. cit.*, p. 61. It is strange to find A. Koyré in agreement with Galileo's atti-

Before considering Galileo's astronomical work it is necessary to clarify the meaning of scientific truth as developed by Grossteste and the Medieval Schoolmen who broke away from Aristotelian concepts of science in the thirteenth century. To them physics or more generally the physical sciences, as such, were no longer concerned with ultimate reality. All questions on primal substances, essences and ultimate causes were relegated to the science of metaphysics. The objective of the physical sciences was henceforth to answer questions on the immediate efficient causes of phenomena. Briefly, the usual order of the process by which such answers were to be determined consists, in the first place, in intelligent observation. The second step is the setting up of hypotheses as a result of inductive thought on these observations. As part of such hypotheses or auxiliary to them, there may be set up a mathematical or mechanical model to simplify and clarify the hypotheses. The third step consists in the execution of experiments (controlled if possible) which test the hypotheses directly or deductions from them. The more experimental evidence acquired in favor of the hypotheses the stronger the hypotheses become. One piece of confirmed experimental evidence which contradicts a hypothesis is sufficient to cause either the complete rejection or the modification of the hypothesis. Finally, a hypothesis which has been proved to be in agreement with experimental evidence or, as more generally happens, a combination of such hypotheses, helps to formulate by logical deduction into an intelligible system or set of laws a series of physical facts which in themselves are seemingly isolated. These laws should be suggestive of further experimentation and expansion in accordance with the deductions which can logically be derived from them; such deductions are usually expressible in mathematical form. Our scientific truths are contained in such laws which, because of the method of derivation, can never be final or absolute but must always remain subject to revision. It is also possible to set up hypotheses on purely intuitional ideas, but the evaluation of all hypotheses depends primarily on the amount of agreement between them and the phenomena which they are supposed to represent as disclosed by experimentation.

The history of scientific progress or man's endeavor to improve his knowledge of nature is marked by the tombstones of abandoned hypotheses. What was thought to be true at one period is later shown, usually as a result of better experimentation, to be either totally at variance with the natural phenomena—in which case the former knowledge must be considered as false—or only partially in agreement with the phenomena—in which case there is proportionally partial truth. A scientific theory is true only in so far as it remains uncontradicted by experimentation. Two or more hypotheses on the same phenomena could, in this sense, be considered as simultaneously true. Since experimentation depends on measurement and measurement of its nature is subject to error, scientific knowledge, as now conceived, cannot aspire to produce the complete truth about any natural phenomena.[15] If two or more hypotheses appear to agree equally well with observable facts or, as the Greeks said, "save the phenomena," a

tude in the case mentioned above. Mathematical deduction of itself is not sufficient to establish the truth of natural phenomena. See his article "Galileo and the Scientific Revolution of the Seventeenth Century," *The Philosophical Review*, July, 1943, p. 347.

[15] " . . . no contingent, hypothetical laws, however wide, can offer an ultimate explanation of concrete facts. Laws are but the expression of the *modus agendi*, the manner of acting, of causes or combinations of causes. And we shall not have fully explained any concrete fact in the universe until we know why the agencies of nature act according to those widest laws—why, for example, matter gravitates, or why life comes only from life, or why natural causes act uniformly." P. Coffey, *The Science of Logic* (New York: Longmans, Green and Co., 1918), Vol. 2, p. 240.

scientist is likely to choose the simpler hypothesis, not because it is more significant of reality but because it more easily leads to new experiment. A simpler hypothesis is not necessarily in better agreement with the actual facts than a more complex one; neither is the simpler mathematical relationship of a physical law necessarily closer to actuality than a more complex one, even though this idea, Pythagorean in origin, has often supplied the necessary impetus to formulate new hypotheses.[16]

In 1609 when Galileo's astronomical work really began he had a choice of three hypothetical systems recently proposed as replacements for the Ptolemaic system. Up to the fifteenth century Europeans, in general, found that the geocentric system, as explained in such popular treatises as the *Sphere* of Sacrobosco, was sufficiently accurate for the needs of the times. But by the sixteenth century matters had considerably changed. The tremendous impetus given to navigation in the previous century had made acute the need for more accurate astronomical data. Developments in surveying and navigation instruments had reciprocally aided in the production of better instruments for astronomical observation. The work of the Portuguese school of navigation, established in 1416, and the astronomical studies of the mid-Europeans under the influence of Cardinal Nicholaus de Cusa, Purbach and Regiomontanus had prepared the way for new developments. Another impetus was given by the need felt by the Catholic Church authorities for a reform of the calendar. It was becoming more and more apparent that the complicated Ptolemaic system was unsatisfactory.

Such were the conditions under which Copernicus undertook to promulgate once more the heliocentric hypothesis of Aristarchus. He was actuated primarily by the desire to provide a simpler mathematical model than that of the eighty-odd spheres necessary for explanation and calculation in the Ptolemaic system. For Copernicus, the heliocentric system represented nothing more than a simplified mathematical hypothesis of thirty-four spherical orbits which would also serve to save the phenomena. For him it was not the result of induction from observational data. He spent many years of his life pondering over this mathematical hypothesis but did little practical observation. Whereas in terms of Aristotelian physics there had been a plausible explanation of the Ptolemaic system, Copernicus was unable to offer any acceptable scientific explanation of his system either in terms of the anti-Aristotelian physics of motion or in terms of the older concepts of motion. For this reason and also on religious grounds Tycho Brahe, the greatest astronomical observer of the ensuing period, rejected the Copernican hypothesis, but since he also saw defects in the Ptolemaic system, he established a compromise system of his own. Kepler, the great computer and enthusiastic follower of Pythagorean number mysticism, who fell heir to the observational data of Brahe was an ardent supporter of the heliocentric system, but found it necessary to change Copernicus' circular orbits into elliptical ones so that the mathematical model would be in better agreement with the calculations he had laboriously worked out from the data of Brahe. All three systems still needed explanation and confirmation in terms of anti-Aristotelian physics. Let us now examine what Galileo actually did with these hypotheses.

Galileo claimed and most of his biographers to date have also claimed that, as a result of his astronomical observations and experiments on motion, he had proved the Aristarchian-Copernican hypothesis to be scientifically true. If this were so he should have been able to produce experimental data either in proof of the two main physical facts implied by the hypothesis, namely, that the earth revolved daily on its own axis and that it revolved annually

[16] For further analysis of scientific truth, see E. F. Caldin, *The Power and Limits of Science* (London: Chapman and Hall, 1949).

round the sun, or at least in proof of deductions from these facts. Such proofs, as given by Galileo, are to be found in his *Dialogo sopra i due Massimi Sistemi del Mondo, Tolemaico e Copernicano.*[17] In 1597 Galileo stated in a letter to Kepler that he was a firm believer in the Copernican system; therefore, *The Two Systems* may justly be considered the result of thirty-three years of research with the objective of establishing the scientific truth of the hypothesis. These proofs as given by Galileo break down into three groups. The first group is in terms of the physics of motion. They consist, in the main, of arguments against Aristotelian ideas on motion which had been bandied across Europe for centuries since the rise of anti-Aristotelian physics. Despite his work on dynamics Galileo was unable to furnish a convincing argument in favor of either motion of the earth because he had failed to bring to completion the theory of impetus. With regard to Galileo's proofs from motion Butterfield points out:

"In his mechanics he was a little less original than most people imagine, since, apart from the older teachers of the impetus theory, he had had more immediate precursors, who had begun to develop the more modern views concerning the flight of projectiles, the law of inertia and the behaviour of falling bodies. He was not original when he showed that clouds and air and everything on the earth—including falling bodies—moved around with the rotating earth, as part of the same mechanical system. . . . His system of mechanics did not quite come out clear and clean, did not even quite explicitly reach the modern law of inertia, since even here he had not quite disentangled himself from obsessions concerning circular motion."[18]

Galileo's second group of proofs are in terms of his telescopic discoveries. From observations of sun-spots he concluded that the sun was revolving on its axis. Parenthetically it may be mentioned that the revolution of the sun on its own axis was not part of the Copernican hypothesis. Galileo discovered that the moons of Jupiter revolved round that planet and explained the phases of Venus in terms of its revolution round the sun. Granted that his explanations of these phenomena were correct, they are nothing but proofs from analogy so far as the motions of the earth are concerned and could not be accepted as proofs from experiment in any system of physical sciences. Galileo knew only too well that these first two groups of proofs would not and could not be acceptable as scientific proofs for the Copernican hypothesis, and therefore, as he says himself, he reserved to the end of his book his clinching argument for the rotation of the earth. This, his main proof, turns out to be nothing more than the erroneous doctrine that the main factor in the production of the tides was the diurnal rotation of the earth.

"I have been induced upon no slight reasons to omit these two conclusions (having made withal the necessary presupposals) that in case the terrestrial Globe be immoveable, the flux and reflux of the Sea cannot be natural; and that, in case these motions be conferred upon the said Globe, which have been long since assigned to it, it is necessary that the Sea be subject to ebbing and flowing, according to all that which we observe to happen in the same."[19]

It should be borne in mind that Galileo arrived at this conclusion despite the fact that Kepler and others had, years before, suggested lunar influences as the main cause of the tides. As an argument against one of these lunar advocates Galileo writes:

"To that Prelate I would say that the Moon moveth every day along the whole Mediterrane, and yet its waters do not rise thereupon, save only in the very extream bounds of it Eastward and here to us at Venice."[20]

Is it then any wonder that the following estimate of Galileo's astronomical work is to be found in the *Encyclopaedia Britannica:*

"The direct services of permanent value which Galileo rendered to astronomy are vir-

[17] G. Galilei, *op. cit.*, Vol. 7, English translation by Thomas Salusbury, *Mathematical Collections and Translations*, Vol. 1 (London: 1661). Hereafter referred to as *The Two Systems.*

[18] H. Butterfield, *op. cit.*, p. 54.

[19] G. Galileo, *op. cit.* (17), Salusbury translation, p. 380.

[20] *Ibid.*, p. 383.

tually summed up in his telescopic discoveries. To the theoretical perfection of the science he contributed little or nothing."[21]

Butterfield's recent conclusion on the value of Galileo's proofs is unequivocal:

"At the end of everything Galileo failed to clinch his argument—he did not exactly prove the rotation of the earth—and in the resulting situation a reader could adopt his whole way of looking at things or could reject it *in toto*—it was a question of entering into the whole realm of thought into which he had transposed the question."[22]

As an astronomer Galileo ignored the work of Kepler whose first and second mathematical hypotheses of planetary motion were made known in 1609, to be followed by a third ten years later. He also ignored the life-time accumulation of observational data of Brahe, and as Dreyer points out:

"In the whole book [*The Two Systems*] there is no allusion whatever to the Tychonic system although it is scarcely too much to say that about the year 1600 nobody, whose opinion was worth caring about, preferred the Ptolemaic to the Tychonic system."[23]

As a matter of historic fact the heliocentric hypothesis was not even completely plausible until Newton had explained it in terms of his synthesis of mathematics, mechanics, gravitation through a hypothetical medium and Kepler's elliptical orbits. Newton's explanation of the solar system appeared in 1687 in his famous *Principia*. The first experimental proof that the earth was revolving on its own axis occurred, more or less accidentally, when the French astronomer Jean Richer found that his pendulum clock, regulated for Paris, lost about two and a half minutes per day when set up at Cayenne, South America, in 1671. The change in the time of the pendulum oscillation could be explained in terms of a revolving, oblate, spheroidal earth flattened at the poles, an idea which both Newton and Huygens had derived by mathematical deduction. The conclusion from Richer's experience was contradicted in the ensuing years by the Cassini surveys, but was later reaffirmed by the more elaborate surveys carried on by the French Academy of Science in Peru, Ecuador and Lapland during the years 1735–43. Experimental proof of the earth's rotation round the sun was a more difficult problem. The first tangible evidence for it came when the astronomer Bradley had observed and calculated the phenomena of aberration of light in 1729.

In view of all this evidence it is not surprising that theologians of the seventeenth century should have been reluctant to give unequivocal consent to Galileo's arguments in *The Two Systems* as proof of the scientific truth of the Copernican hypothesis. This is not the place to reiterate at length the facts leading to Galileo's condemnation by an ecclesiastical commission. If any reader wishes a clear, concise account of the whole affair he will find it in an article by Father Conway.[24] If he wishes for more details he has but to delve into the works of Von Gebler[25] and Favaro.[26] Here it will suffice to state that Galileo was twice tried by Church tribunals, the first time at his own request. As a result of the 1616 trial he was admonished privately by Cardinal Bellarmine, at the request of Pope Paul V, to abandon his opinions on the heliocentric system. To this admonition Galileo aquiesced. In 1633 he was publicly condemned after the publication of *The Two Systems*. The primary

[21] Agnes M. Clerke, "Galileo Galilei," *Encyclopaedia Britannica* (14th ed., 1946), vol. 9, 980.

[22] H. Butterfield, *op. cit.*, p. 54.

[23] J. L. E. Dreyer, *History of the Planetary Systems from Thales to Kepler* (Cambridge University Press, 1906), p. 416.

[24] Pierre Conway, O.P. "Aristotle, Copernicus, Galileo," *The New Scholasticism*, Vol. 23, Nos. 1 and 2, 1949.

[25] Carl Von Gebler, *Galileo Galilei und die römische Curie* (Stuttgart: 1876). English translation by Mrs. G. Sturge, London, 1879. This was the first historical research to make use of all the Vatican records relative to Galileo's trials.

[26] See note 3. This work, in twenty volumes, contains all Galileo's scientific treatises and available letters together with numerous other letters and documents relative to Galileo's work.

reason for the first trial was Galileo's provocative attitude on the question of interpretation of the Scriptures. Professor Carnahan's statement that "Theologians discovered that their doctrines and Galileo's science were not in agreement" is a strange twist to put on the controversy, but again, oft-repeated like the Tower of Pisa story. Far nearer the truth is the statement by Strong in his study of the development of science in the sixteenth and seventeenth centuries:

> "The furore raised against Galileo in connection with the Copernican hypothesis cannot be taken to be the attitude of the Church against the mathematical-physical sciences in general. The pursuit of physical science involved Galileo in controversies with men opposed to his physical views, but this was not a clash over the religious consequences involved in astronomical questions."[27]

The results of both trials show that the examiners were satisfied that Galileo's proofs were insufficient to raise the Copernican hypothesis to the level of a scientific truth. With these decisions the calmer judgments of modern scientists must corroborate.

If Galileo had presented his scientific theories without being so arrogant as regards the positiveness of his proofs, and had avoided being drawn into the trap of Scriptural controversy, set up by some of his more wily opponents after the publication in 1613 of his *Istoria e Dimostrazioni intorno alle Macchie Solari*[28] in which he first publicly stated his decided Copernican views, the trial of 1616 would probably never have occurred. And if there had been no directive from the Congregation of the Index, as a result of the 1616 trial, that Copernicus' book was to be banned until its hypothetical nature was made clearer—which correction was carried out within a short while—the inquiry of 1633 before the Inquisition would have been unnecessary. At the second trial Galileo was accused of breaking his promise of 1616, since any intelligent person could see that *The Two Systems* in no way favored the hypothetical nature of the Copernican system, despite the ruse which Galileo used in trying to cover up his real opinions on the subject. That there are reasonable grounds for this assumption may be seen from the attitude of Cardinal Bellarmine who was regarded as anti-Copernican. In a letter written to the Carmelite, Father Foscarini, an advocate of Copernicanism, shortly before the 1616 trial he states:

> "I say that when there shall be a real demonstration that the sun stands in the center of the universe and the earth revolves around it, it will then be necessary to proceed with great consideration in explaining those passages of Scripture which seem to be contrary to it, and rather to say we do not understand them, than to say that a thing which is demonstrated is false. But I will not believe that there is such proof until it is shown to me; nor is it the same thing to show that the phenomena are saved by assuming that the sun is in the center and the earth revolves round it, and to show that in reality the sun is in the center and the earth revolves."[29]

These are the words of an outstanding theologian of the period but surely, no one can read into them that he is trying to direct the course of scientific investigation so as to make it conform to a religious belief; rather is it the statement of a careful scientist. Similar opinions can be found in the letters of many prominent Catholic ecclesiastics of the period.

Besides, since the publication of Copernicus' *De Revolutionibus Orbium Coelestium* in 1543 there had been no opposition to the heliocentric hypothesis on the part of the Catholic Church; whatever opposition existed came from individuals of various religious persuasions. In fact many of the higher Catholic clergy were known to favor it openly. Was it not through the instrumentality of the Catholic clergy that the Aristarchian heliocentric system had been revived, first by Cardinal Nicholaus de Cusa in his *De Docta Ignorantia* (*c.* 1450) and again, nearly a century later, by

[27] E. W. Strong, *Procedures and Metaphysics* (California University Press, 1936), p. 137.

[28] G. Galilei, *op. cit.*, Vol. 4.

[29] D. Berti, *Copernico e le vicende del sistema Copernicano in Italia* (Rome, 1876), p. 123.

the canon Copernicus? When Galileo visited Rome in 1611 to demonstrate his telescopes he was enthusiastically received by the Church authorities and especially by the future Pope Urban VIII. It is also important to notice that, despite the 1616 trial, Galileo's ecclesiastical supporters did not abandon him; he even became the recipient of a life pension from his very personal friend Urban VIII in 1624, a reward for his scientific endeavors, setting the precedent, in modern times, for scientific stipends! There was no action on the part of the Catholic Church during this period to show that it was in opposition to scientific progress. Any unbiased history of the period shows the opposite to be true. Nevertheless Galileo knew that, unless he could produce the experimental evidence, anything beyond a hypothetical statement of the Copernican system would run counter to the then accepted interpretation of the Scriptures. He also knew that there was nothing in Catholic theology or tradition against the Church's sanctioning a change in interpretation if necessity arose. What really led to Galileo's troubles was the method he adopted to bring about this change. Like many another highly gifted man who lacked tact he found himself before long antagonizing people who were formerly his best friends. If he had approached the matter cautiously and with less demand that his reasoning be accepted as demonstrative, which it was not, Cardinal Bellarmine and others would undoubtedly have used their influence to allow Galileo to pursue his astronomical work unmolested by any clerical interference, and the Church authorities would have been his best supporters if he had ever reached the stage of producing what the Cardinal called "a real demonstration," that is an actual proof by experimentation of some deduction from the hypothesis.

It is often assumed that the philosophical system of Aristotle and St. Thomas Aquinas was irreconcilable with the researches undertaken by Galileo. But just as Aristotle was admittedly more realistic in certain respects than Galileo, so also St. Thomas might have given him timely—and modern—advice on the necessity of distinguishing between a scientific hypothesis and a scientific demonstration, as the following excerpt shows:

> "The assumptions made by the astronomers are not necessarily true. Although these hypotheses seem to be in agreement with the observed phenomena we must not claim that they are true. Perhaps one could explain the observed motion of the celestial bodies in a different way which has not been discovered up to this time."[30]

A recent statement of Whittaker on the influence of the work of Brahe and Kepler is also apposite here:

> "At this point it may be observed that, while the Scholastic cosmology was thereby completely disproved and overthrown, there was nothing in the new methods and discoveries that was inherently irreconcilable with the Scholastic metaphysics; the whole of Tycho's and Kepler's work might conceivably have been absorbed into the philosophy of the Schoolmen by a peaceful and conservative revolution. If this had happened, we in the twentieth century should have been spared the necessity of readjusting our position by a movement back towards Aristotelianism."[31]

But this did not happen because of the lack of true Aristotelianism at the time. Statements found in many of our popular histories of science to the effect that the experimental work of Galileo was completely at variance with the methods of the Aristotelian philosophers of the Italian universities of the seventeenth century are a calumny on Aristotle. A professor of natural philosophy who would refuse to look through a telescope, as some of Galileo's scientific opponents did, should not for that reason be classified as a follower of Aristotle. Not only was Aristotle one of the greatest exemplars of the use of deductive logic but, and this is too often forgotten, he was also well aware of the value and pitfalls of induction, as a read-

[30] St. Thomas Aquinas, *Commentary on Aristotle's De Caelo II*, lectio 17, no. 2, (Rome, 1886).

[31] E. Whittaker, *op. cit.*, p. 251.

ing of the first section of his *Metaphysics* will show. His insistence on the necessity of observation and experiment may be seen from what he has to say of those who

"... had certain predetermined views and were resolved to bring everything into line with them. ... As though some principles did not require to be judged from their results, and particularly from their final issue! And that issue, which in the case of productive knowledge (i.e. in the case of art) is the product, in the knowledge of nature is the unimpeachable evidence of the senses as to each fact."[32]

Why then do we continue to associate the name of Aristotle with a system of natural philosophy which had become atrophied by lack of experimentation? The conflict between Galileo and his opponents in the realm of natural philosophy was caused by an absence of true Aristotelian principles and this was in no way confined to the members of one religious group. Even Galileo admitted that Aristotle would undoubtedly change his ideas about certain physical phenomena if he were then living and could have participated in the experimental work being carried on.

Milder judgments on the decisions of the Inquisition in 1633 than that implied by Professor Carnahan have long since been passed by men of high standing in the scientific world who cannot be accused of bias toward the Catholic Church. Thomas Huxley, in a letter to Professor Mivart in 1885 writes:

"I have looked into the matter [the Galileo controversy] when I was in Italy, and I have arrived at the conclusion that the Pope and the College of Cardinals had rather the best of it."[33]

The following from the latest edition of Dampier shows the change in attitude from the unwarranted statements of condemnation of the Catholic Church to be found in so many of the nineteenth century histories of science, and unfortunately still appearing from time to time.

"In spite of Whewell's clear and fair account of the incident some more recent writers have made too much of the persecution of Galileo for his Copernican views. As Whitehead [A. N.] says 'In a generation which saw the Thirty Years War and remembered Alva in the Netherlands, the worst that happened to men of science was that Galileo suffered an honorable detention and a mild reproof, before dying peacefully in his bed.' "[34]

Butterfield concluding his essay on the science of this period says:

"Aristotelian physics were clearly breaking down, and the Ptolemaic system was split from top to bottom. But not until the time of Newton did the satisfactory alternative system appear. ... The long existence of this dubious, intermediate situation brings the importance of Sir Isaac Newton into still stronger relief. We can better understand also, if we cannot condone, the treatment which Galileo had to suffer from the Church for a presumption which in his dialogues on *The Two Principal World-Systems* he had certainly displayed in more ways than one."[35]

The man who, in Professor Carnahan's words, had to live "the last ten years of his life under the sternest requirement to say nothing, write nothing, think nothing of which entrenched authority does not approve" produced during that time the treatise on which his proper claim to fame rests, that is, the *Two New Sciences*, a compendium of his life's work on mechanics and allied topics. He also had time to work in conjunction with Torricelli and Viviani, and to write numerous letters on various scientific topics.

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[32] Aristotle, *op. cit.*, De Caelo III, 306[a].

[33] Thomas H. Huxley, *Life and Letters*, (ed. L. Huxley), London, The Macmillan Co., 1900, Vol. 2, p. 122. It is interesting to compare this with Huxley's statement on the same subject in his essay "Descartes' Discourse on Method" written in 1870.

"At that time, physical science suddenly strode into the arena of public and familiar thought and openly challenged not only Philosophy and the Church, but that common ignorance which often passes by the name of Common Sense. The assertion of the motion of the earth was a defiance of all three, and Physical Science threw down her glove by the hand of Galileo. ... Charity children would be ashamed not to know that the earth moves; while the Schoolmen are forgotten; and the Cardinals—well, the Cardinals are at the Oecumenical Council, still at their old business of trying to stop the movement of the World." *Collected Essays*, New York, D. Appleton and Co., 1894, Vol. 1, pp. 179, 180.

[34] Dampier, W. C., *A History of Science*, Cambridge University Press, 1949, p. 113.

[35] Butterfield, H., *op. cit.*, p. 55.

Evaluation of Determinants

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The proposed method of evaluation of determinants, however unorthodox it may seem, has, for some unknown reason, escaped the attention of authors and teachers of algebra, although references to it (and complete developments) have appeared in the literature on the subject.[1] As a method for the evaluation of determinants it is general and it is, in some respects, more effective due to its simplicity. This proposed method is not offered with the view of supplanting and replacing the traditional method of evaluating determinants by means of direct expansion or by expansion by means of minors. However, it may appeal to some classroom teachers as a process which contains elements of simplicity.

The development of this method utilizes the fundamental properties of determinants, namely: (*a*) The multiplication of a row (or of a column) by some number is equivalent to the multiplication of the determinant by that number, and (*b*) The addition to a column or row (or the subtraction from a column or row) of another column (or row) or of a multiple of a column (or row) term by term leaves the numerical value of the determinant unchanged.

Consider the determinant

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \cdots m_1 & n_1 \\ a_2 & b_2 & c_2 \cdots m_2 & n_2 \\ a_3 & b_3 & c_3 \cdots m_3 & n_3 \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ a_n & b_n & c_n \cdots m_n & n_n \end{vmatrix} \quad [1]$$

Multiply each term of the first column of the determinant by the first term of the second column with its sign changed, that is by $-b_1$. Multiply each term of the second column by the first term of the first column. We then have two columns:

$$\begin{matrix} -a_1b_1 & a_1b_1 \\ -a_2b_1 & a_1b_2 \\ -a_3b_1 & a_1b_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ -a_nb_1 & a_1b_n \end{matrix}$$

Add these two columns term by term. We have then, observing that

$$a_1b_1-a_1b_1=0,$$

a single column

$$\begin{matrix} 0 \\ a_1b_2-a_2b_1 \\ a_1b_3-a_3b_1 \\ \cdot \\ \cdot \\ a_1b_n-a_nb_1 \end{matrix}$$

Perform the same operations on the second and third columns. That is, multiply each term of the second column by the first term of the third column with its sign changed, that is by $-b_3$. Multiply each term of the third column by the first term of the second column. Then add the product-columns term by term, and note that

$$b_1c_1-b_1c_1=0.$$

We then obtain another column

[1] Aaron Bakst. "A Modification of the Computation of the Multiple Correlation and Regression Coefficients by the Tolley and Ezekiel Method," *Journal of Educational Psychology*, Vol. XXII (1931), pp. 629–635.

T. Muir. *Theory of Determinants*, Vol. IV. (London: The Macmillan Co., Ltd., 1923), p. 396.

T. Muir and W. H. Metzler. *Theory of Determinants*. (New York: Longmans, Green and Co., 1933), pp. 63–65.

$$0$$
$$b_1c_2-b_2c_1$$
$$b_1c_3-b_3c_1$$
$$\cdot$$
$$\cdot$$
$$b_1c_n-b_nc_1$$

Continuing the same operations on the remaining columns of the determinant D, we obtain a series of columns. The only column of the determinant D which will remain unchanged is the column on the extreme right, that is, the column with the terms, $n_1, n_2, n_3, \cdots, n_n$. We can then write a new determinant whose terms will be the elements of the columns obtained as indicated above. Also note that this new determinant will be equal to the product of the determinant D and

$$(-1)^{n-1}b_1c_1 \cdots n_1.$$

Then

$$(-1)^{n-1}b_1c_1 \cdots n_1D = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & n_1 \\ a_1b_2-a_2b_1 & b_1c_2-b_2c_1 & c_1d_2-c_2d_1 & \cdots & m_1n_2-m_2n_1 & n_2 \\ a_1b_3-a_3b_1 & b_1c_3-b_3c_1 & c_1d_3-c_3d_1 & \cdots & m_1n_3-m_3n_1 & n_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1b_n-a_nb_1 & b_1c_n-b_nc_1 & c_1d_n-c_nd_1 & \cdots & m_1n_n-m_nn_1 & n_n \end{vmatrix} \quad [2]$$

The determinant [2] is expanded by minors (taking them along the first row). This results in a determinant

$$(-1)^{n-1}b_1c_1 \cdots n_1D = (-1)^{n-1}n_1 \begin{vmatrix} a_1b_2-a_2b_1 & b_1c_2-b_2c_1 & c_1d_2-c_2d_1 & \cdots & m_1n_2-m_2n_1 \\ a_1b_3-a_3b_1 & b_1c_3-b_3c_1 & c_1d_3-c_3d_1 & \cdots & m_1n_3-m_3n_1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_1b_n-a_nb_1 & b_1c_n-b_nc_1 & c_1d_n-c_nd_1 & \cdots & m_1n_n-m_nn_1 \end{vmatrix} \quad [3]$$

The determinant [3] consists of $(n-1)$ rows and $(n-1)$ columns.

Note that every element of the determinant in [3] represents an expansion of a two-rowed determinant. This fact permits us to rewrite the expression [3] as follows:

$$D = \frac{1}{b_1c_1d_1 \cdots m_1} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} & \cdots & \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} & \cdots & \begin{vmatrix} m_1 & n_1 \\ m_3 & n_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_4 & c_4 \end{vmatrix} & \begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix} & \cdots & \begin{vmatrix} m_1 & n_1 \\ m_4 & n_4 \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_n & c_n \end{vmatrix} & \begin{vmatrix} c_1 & d_1 \\ c_n & d_n \end{vmatrix} & \cdots & \begin{vmatrix} m_1 & n_1 \\ m_n & n_n \end{vmatrix} \end{vmatrix} \quad [4]$$

The expression [4] indicates the procedure which can be employed in the process of the evaluation of the determinant. This process may be summarized as follows:

(1) Within the determinant which is being evaluated, construct two-rowed determinants in which the first rows will consist of the terms of the first row of the determinant which is being evaluated, while the second rows will consist of the corresponding terms which are in the rows, starting with the second, in a manner indicated in the expression [4].

For example let the determinant which is to be evaluated be

$$\begin{vmatrix} 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 3 \\ 3 & 2 & 4 & 5 \\ 1 & 4 & 2 & 3 \end{vmatrix}. \qquad [5]$$

Then the two rowed determinants are obtained as follows:

$$\begin{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 4 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} \end{vmatrix}. \qquad [6]$$

(2) Divide the new determinant thus obtained by the product of the terms in the first row of the original determinant omitting the first and the last terms. Thus, the determinant [5] finally becomes

$$\frac{1}{3 \cdot 2} \begin{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 4 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} \end{vmatrix}. \qquad [7]$$

It must be carefully observed that none of the factors of the product of the terms of the first row thus obtained is equal to zero. If one (or several) terms in the first row (except the first and the last) is zero, then the method of evaluating the determinant just described will fail, unless certain measures are taken in order to eliminate these zero-terms. These measures may involve the interchanging of rows or columns or the addition of rows or columns, or the addition of multiples of rows or columns.

It should be also noted that the above method of evaluating a determinant applies to the first column also. Thus the determinant [5] may be rewritten as

$$\frac{1}{2 \cdot 3} \begin{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} \end{vmatrix}. \qquad [8]$$

(3) Evaluate the two-rowed determinants which have been thus obtained. This will result in a determinant which contains one row and one column less than the original determinant.

Evaluating the determinants in the expressions [7] and [8] we obtain:

(*a*) For the expression [7]

$$\frac{1}{3 \cdot 2} \begin{vmatrix} -2 & -5 & 1 \\ -7 & 8 & -10 \\ 1 & -2 & -4 \end{vmatrix}.$$

(*b*) For the expression [8]

$$\frac{1}{2 \cdot 3} \begin{vmatrix} -2 & -3 & -7 \\ -8 & 5 & 1 \\ 10 & 2 & 4 \end{vmatrix}.$$

(4) Continue the same processes as indicated in (1)–(3) above.

We have then the following evaluations:

(*a*) For the determinant in expression [7]

$$\frac{1}{(-5)\cdot 3\cdot 2}\begin{vmatrix}\begin{vmatrix}-2 & -5\\ -7 & 8\end{vmatrix} & \begin{vmatrix}-5 & 1\\ 8 & -10\end{vmatrix}\\ \begin{vmatrix}-2 & -5\\ 1 & -2\end{vmatrix} & \begin{vmatrix}-5 & 1\\ -2 & -4\end{vmatrix}\end{vmatrix} = -\frac{1}{30}\begin{vmatrix}-51 & 42\\ 9 & 22\end{vmatrix} = 50$$

(*b*) For the determinant in expression [8]

$$\frac{1}{(-8)\cdot 2\cdot 3}\begin{vmatrix}\begin{vmatrix}-2 & -3\\ -8 & 5\end{vmatrix} & \begin{vmatrix}-2 & -7\\ -8 & 1\end{vmatrix}\\ \begin{vmatrix}-8 & 5\\ 10 & 2\end{vmatrix} & \begin{vmatrix}-8 & 1\\ 10 & 4\end{vmatrix}\end{vmatrix} = -\frac{1}{48}\begin{vmatrix}-34 & -58\\ -66 & -42\end{vmatrix} = 50.$$

As stated above, this method of evaluating determinants will fail if some of the terms in the first row (or in the first column), except the first and the last, are zero. However, as indicated, this may be avoided by means of interchanging rows or columns or by means of the addition (or subtraction) of rows or columns (or of their multiples). The following example will illustrate this procedure. Let us evaluate the following determinant

$$\begin{vmatrix}2 & 0 & 3\\ 0 & 4 & 0\\ 1 & 0 & 2\end{vmatrix}.$$

Adding the first column to the second we obtain

$$\begin{vmatrix}2 & 2 & 3\\ 0 & 4 & 0\\ 1 & 1 & 2\end{vmatrix}$$

and the evaluation of the determinant thus becomes possible. We have

$$\frac{1}{2}\begin{vmatrix}\begin{vmatrix}2 & 2\\ 0 & 4\end{vmatrix} & \begin{vmatrix}2 & 3\\ 4 & 0\end{vmatrix}\\ \begin{vmatrix}2 & 2\\ 1 & 1\end{vmatrix} & \begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\end{vmatrix} = \frac{1}{2}\begin{vmatrix}8 & -12\\ 0 & 1\end{vmatrix} = 4.$$

Interchanging columns we have

$$-\begin{vmatrix}0 & 2 & 3\\ 4 & 0 & 0\\ 0 & 1 & 2\end{vmatrix} = -\frac{1}{2}\begin{vmatrix}\begin{vmatrix}0 & 2\\ 4 & 0\end{vmatrix} & \begin{vmatrix}2 & 3\\ 0 & 0\end{vmatrix}\\ \begin{vmatrix}0 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\end{vmatrix} = -\frac{1}{2}\begin{vmatrix}-8 & 0\\ 0 & 1\end{vmatrix} = 4.$$

A series of TV shows "Understanding Numbers: Their History and Use" is being presented on Detroit's WWJ-TV station on Sundays at 1:00 P.M., EST. **Professor Phillip S. Jones** of the **University of Michigan** is directing these presentations, the first of which was televised on February 24. This is one of the "Telecours" series sponsored by the University of Michigan and it will run for seven consecutive weeks.

Influence of Research on Instruction in Arithmetic

J. Wayne Wrightstone

Director of Educational Research, Board of Education of the City of New York

Introduction

Research on instruction in arithmetic has influenced and continues to influence teaching procedures and the content of arithmetic at various grade levels. The influence of research operates through individual teachers and supervisors who keep informed about investigations in arithmetic. It operates, also, through arithmetic courses of study and textbooks which translate research findings on arithmetic content and procedures into suggestions for classroom activities and methods. It is recognized that widespread adoption of research findings in arithmetic instruction is a slow and gradual process, but the evidence is clear that research alters current practices. Since the early days of systematic research on arithmetic in social usage, as reported by Wilson and others (15),* content of the curriculum has progressively changed. The influence of selected research studies on instruction in arithmetic is summarized in this paper.

Research on Arithmetic in Grades 1 and 2

Both the older and newer studies of the arithmetic knowledge of pre-school children and of children in grades 1 and 2 have influenced modern practices in the classroom, courses of study and textbooks. Early investigations by MacLatchy (14) and Woody (26) indicate the average pupil's knowledge of counting, measuring and number processes, before formal school instruction in arithmetic is begun. These studies reveal that, even before children enter school, many of them have attained considerable ability in counting, in reading simple numbers, in telling time, in knowing the common units of measurements and in dealing with simple exercises in addition and subtraction. More recently Mott (17) investigated the knowledge of number concepts possessed by pre-school and kindergarten children and found this knowledge ranged from no knowledge beyond one, two or many to counting through the hundreds. The evidence is clear, therefore, that children do bring a range of arithmetic knowledge to the first grade. The problem is to adapt the curriculum to pupil needs and abilities.

Formal versus informal arithmetic instruction in grades 1 and 2 has been the topic of several studies. The Committee of Seven studies, as reported by Washburne (23), showed that the average child in grades 1 and 2 can readily master the basic addition and subtraction combinations under conditions of formal instruction. On the other hand, studies by Brownell (3) and Wilson (25) indicate the learning of number combinations under more informal conditions, for example, in connection with the activity units, provides equal facility of achievement without formally organized practice or drill work. Tests at succeeding grade levels reveal a rapid increase in knowledge of combinations in addition and subtraction from grades 1 to 3, even when no formal instruction in numbers is given.

Strong evidence, therefore, leads to the conclusion that formal work on number processes can be delayed until at least the second grade, provided steps are taken in the first two grades to provide a series of informal but well organized activities and units of meaningful number situations which utilize arithmetic concepts in the daily affairs of children. These research findings have influenced cities, for example, Philadelphia, New York, Chicago and Los Angeles to formulate courses of study in the learning of arithmetic for the primary grades that begin with non-numeri-

* Numbers in parentheses refer to the bibliography at the end.

cal experiences and advance to number concepts, number symbols, number factors, and number operations. The newer textbooks and teachers' manuals, as well as courses of study, follow this pattern of development.

Methods of Instruction in Systematic Arithmetic

Research on methods of instruction in systematic arithmetic is so extensive that certain aspects only have been selected for discussion and illustration. It may be of interest, however, to refer to an article by Stevens (20) in which five major trends in teaching arithmetic are identified as: (a) increase in emphasis given to concept building, (b) emphasis on individual rather than mass instruction, (c) delay in the introduction of abstract work, (d) increase in oral arithmetic, and (e) recognition and acceptance of immature ways of dealing with number situations. The following topics are selected on bases, or criteria, different from those used by Stevens. Many of these topics represent issues upon which experts have expressed different points of view.

Individual Differences

Many studies have shown the range of arithmetic abilities among children in a classroom. This raises the problem of how appropriate differentiation and individualization may be provided. In a large city system, even with a relatively homogeneous fifth grade group, the range may cover from the third through the eighth grade level of arithmetic ability. At the first grade level Mott (17) has shown a wide range in number, or arithmetic, ability possessed by pre-school and kindergarten children before they entered first grade. Wilburn (24) has reported on a method of self-instruction to assist pupils to use a ten in subtracting. His use of ten in subtraction was defined as the rearranging of a group of tens and ones so that a larger group of ones than exists in the original group can be removed.

To provide for differences in the rate of learning number process, materials have been devised that make it possible for pupils to progress at different rates in the mastery of computational skills as well as number concepts and meaning. Work books by various publishers are available and may be used in a manner so as to provide for individual differences in rates of learning. These materials are widely recommended in courses of study.

Difficulty of Combinations

Despite some differences on the detailed order of difficulty, most investigators have found that some combinations are more difficult than others for children to master. Washburne and Vogel (22) studied the difficulty of number combinations and checked their results against the study of Clapp (5). They found that, regardless of the method of teaching and regardless of the method of testing, that certain number combinations are inherently more difficult than others. These difficulties tended to disappear in the case of addition above grade 3, but continued to persist in the other three processes, especially in multiplication. In a study by Clapp (5) and a study by Ruch (19) it was found that at the end of grade 3, combinations in addition and subtraction are practically mastered. Studies of pupils' knowledge of multiplication and division show a progressive increase in both rate and accuracy of work from grades 3 through 8. Most pupils master these combinations by the end of grade 5 but there is a slight and gradual improvement in accuracy after grade 5 that continues until grade 8. These studies have influenced practice in arithmetic instruction as revealed in an analysis of courses of study and in textbooks; hence, modern instruction recognizes and attempts to deal with the difficulty of various combinations and processes.

Formal vs. Informal Instruction

Formal versus informal arithmetic instruction has been a topic of research, especially in grades 3 through 8. Studies by Hanna (8), Harap and Mapes (9, 10), and

Wrightstone (27) have shown that processes in whole numbers, fractions and decimals can be learned through practical uses in carefully selected activities adapted to the level of development and interests of the pupils. These activities, however, must involve the need for performing computations involving these processes. While the research data on this problem are inconclusive because previous arithmetic instruction of pupils was not carefully controlled, it would appear that it is desirable to present number processes in connection with meaningful experiences even though the arithmetic program is planned to carry on subsequently systematic practice with prepared materials. This compromise on formal and informal experiences is widely represented in current newer courses of study.

Use of Crutches

Another moot problem in arithmetic instruction is the use of crutches. Brownell (2) has carried on investigations, one of which involved the use of a crutch in adding proper fractions and another which involved borrowing in subtraction. In these studies groups using the crutch did approximately as well as the non-crutch group. Both groups made the same kinds of errors. Thus the crutch was not effective in reducing the number or kinds of errors. An interesting finding was that the group using the crutch tended to drop it as soon as the crutch had served its purpose as an aid to learning in the early stages of the development of a process.

This finding has influenced practice in teaching arithmetic so that in the more recent courses of study as well as the more recent textbooks in arithmetic the crutch is no longer taboo. The use of the crutch is recommended when it facilitates understanding and computation in the early stages of learning a process.

Subtraction Methods

Another problem in instruction is the decomposition versus additive methods of teaching subtraction. The studies on the relative effectiveness of these two methods shows no large difference in favor of either method. The majority of studies, however, tend to favor the decomposition, or "take away" method. In view of this fact and the fact that the decomposition method provides for a more meaningful approach to learning arithmetic, it is the method favored in most present day courses of study and most textbooks on arithmetic.

Trial Divisors

Another topic of debate and study is the use of trial divisors. Evidence from various studies is somewhat inconclusive. Osburn (18) made a comprehensive series of studies and for a two digit division and one digit quotient recommended that the apparent method be taught to beginners. Morton (16), on the basis of his studies, recommended the increase-by-one method for two digit divisors ending in 6, 7, 8 and 9. On the basis of special studies Grossnickle (6) favors the apparent method. At the present time and in conformity with the inconclusive findings of research, each of these methods will be found represented not only in current courses of study but also in current textbooks.

Meanings in Arithmetic

The meaning theory requires that arithmetic should be intelligible, sensible and useful to the learner, rather than mechanical. It stresses the ability to think through a quantitative situation and to arrive at sound conclusions. This involves an understanding of the nature of the number system and especially of the numbers and relationships which serve to make it an instrument of thinking. It involves a command of number symbolism. It involves a command of fundamental processes. It involves the ability to relate grouping arrangements for specific social situations with principles of mathematical grouping studied earlier.

There is a body of evidence slowly accumulating which indicates that generalization of ideas and processes which help

in transfer of knowledge in arithmetic are important for classroom instruction. Among the early investigations on meaning and generalization in arithmetic are the studies by McConnell and Thiele. McConnell (13) found that second grade pupils who learned addition and subtraction combinations by procedures which emphasize discovery, organization and generalization versus mixed practice and specific drill achieved better scores on tests which involved transfer of training to untaught processes. Thiele (21) compared the learning of 100 addition factors by methods of specific repetition and meaningful generalization. At the end of the experiment Thiele administered a transfer test composed of 30 addition examples, each of which contained one addend larger than ten. The results showed a difference in favor of the pupils who had used the generalization method. More recently Brownell (1) conducted a careful experiment in third grade teaching of subtraction. The decomposition and equal-additions methods were taught mechanically and rationally. Brownell found that both methods could be made meaningful to third grade pupils; however, if the decomposition method is taught, it should be taught rationally, but the equal addition method could be taught either mechanically or rationally. Transfer accuracy was superior for the decomposition method taught rationally.

From the evidence of these studies and related discussions there has been a definite trend both in courses of study in large cities as well as in more recent textbooks to introduce more meaningful approaches or procedures to the teaching of arithmetic.

Arithmetic Vocabulary

From a behavioral point of view, vocabulary and concepts in arithmetic are closely related to meaning. Buswell (4) has shown that in general there is a growth in understanding by children of such basic words in arithmetic vocabulary as average, estimate, and century. This rate of growth improves slowly from grade to grade. In the case of some words, such as acre, perimeter and rectangle, there is little growth. These differences and the wide variations in knowledge of pupils from school to school are assumed to be indicative of difference in emphasis placed upon vocabulary and concept development. Grossnickle (7) also has made studies of the relative difficulty of various groups of items about business at the eighth grade level and has concluded that there is wide variation in achievement of pupils from school to school. The studies in this area are inconclusive and the corresponding items and practices in courses of study and textbooks vary considerably, showing the same inconclusive trends.

Problem Solving in Arithmetic

Studies of problem solving have been reviewed critically and summarized by Johnson (11). He has organized these studies under such headings as causes of difficulty, improvement of problem solving and the like. In spite of the fragmentary, inconsistent and inconclusive nature of the research findings, Johnson expresses optimism about the possibilities of further research on problem solving. This much is clear from the present studies on problem solving. There is a high degree of correlations between the mental ability of the pupils as measured by verbal intelligence tests and problem solving as measured by current arithmetic tests.

The indefinite nature of research findings on problem solving is represented in the practices as reflected in courses of study and in textbooks. There is an emphasis on problems close to children's experiences but the emphasis frequently found on formal analysis of problems is based on faith and opinion rather than upon research. It is clear that much more fundamental research needs to be done on problem solving before positive guides to action can confidently be supplied to teachers.

Research on problem solving has been fairly extensive, but the results are of negative rather than positive value for the

classroom teacher. As reported in the Encyclopedia of Educational Research (15) formal step analysis is of little value.

These are tentative findings from research: (*a*) Experience in real situations increases the ability to solve problems. Compared to book study alone, pupils of comparable intelligence do significantly better work in problem solving when they have had actual experience with the problem situations. The higher the intelligence of the pupil, however, the less is experience a necessary factor for successful problem solving. (*b*) Extraneous data in a problem cause confusion at all levels of intelligence, but especially for pupils of low intelligence.

Mental Age, Readiness, and Grade Placement

Washburne (23) and the Committee of Seven carried on extensive studies to determine whether or not there are minimum mental age or readiness levels at which it is most efficient for a child to learn various arithmetic processes. The committee concluded that there are stages in a child's mental growth before which it is generally ineffective to teach various arithmetic processes. They recommend, for example, that the easier addition facts should not be taught until a child has reached a minimum mental age of 6 years 5 months. For the harder subtraction facts they suggest a minimum mental age of 7 years 8 months; for multiplication facts a mental age of 10 years 2 months; decimals 10 years 11 months; meaning of fractions 11 years 7 months; and short division 11 years 4 months. These studies have been criticized and defended, but certainly an analysis of grade placement of processes in recent courses of study and textbooks would indicate there is a relationship between the findings and current practices. At least some of the credit for this trend may be assessed against the studies.

Summary

It is reasonable to conclude that research on arithmetic has influenced and will continue to influence teaching procedures and the content of arithmetic. Examples of the application of research findings have been cited to support current practices. They are: (*a*) stressing informal but meaningful number experiences in grades 1 and 2 (*b*) methods of instruction in systematic arithmetic; (*c*) individualization of instruction; (*d*) difficulty of certain combinations; (*e*) a fusion of systematic and informal instruction above grade 3; (*f*) the judicious use of crutches; (*g*) preference for the decomposition method of subtraction; (*h*) inconclusive findings on trial divisors; (*i*) emphasis on the meaning theory in arithmetic instruction; (*j*) inconclusive trends in teaching arithmetic vocabulary; (*k*) the need for more definitive research on problem solving; and (*l*) recent changes in the grade placement of arithmetic processes.

The application of research findings is evident in current courses of study, textbooks, and classroom instruction. As Le Baron (12) points out in his study, mature and successful teachers are far from complete agreement with or acceptance of current research findings and judgments of experts. The gains for research, however, are notable and should gradually exert increased influence on instruction in arithmetic.

Bibliography

1. Brownell, W. A. "An Experiment on Borrowing in Third Grade Arithmetic." *Journal of Educational Research*, 41: 161–71, (January 1947).
2. Brownell, W. A. "Borrowing in Subtraction." *Journal of Educational Research*, 33: 415–24, (February 1940).
3. Brownell, W. A. "Readiness and the Arithmethic Curriculum." *Elementary School Journal*, 38: 344–55; (January 1938).
4. Buswell, G. T. *The Vocabulary of Arithmetic.* (Supplementary Educational Monograph, No. 28) Chicago: University of Chicago Press, 1931.
5. Clapp, F. *The Number Combinations, Their Relative Difficulty, and the Frequency of Their Appearance in Textbooks.* Madison, Wisconsin: University of Wisconsin, 1924.
6. Grossnickle, F. E. "An Experiment in Division with a Two Figure Divisor." *Elementary School Journal*, 37: 668–77, (May 1937).
7. Grossnickle, F. "Concepts in Social Arithmetic for the Eighth-Grade Level." *Journal*

of Educational Research, 30: 475–88, (March 1937).

8. Hanna, Paul R. "Opportunities for the Use of Arithmetic in an Activity Program." *Tenth Yearbook of the National Council of Teachers of Mathematics*, New York: Bureau of Publications, Teachers College, Columbia University, 1935. Pp. 85–120.
9. Harap, H. L., and Mapes, E. "The Learning of Decimals in an Arithmetic Activity Program." *Journal of Educational Research*, 29: 686–93, (May 1936).
10. Harap, H. L. and Mapes, E. "The Learning of Fundamentals in an Arithmetic Activity Program." *Elementary School Journal*, 34: 515–26, (March 1934).
11. Johnson, H. C. "Problem-Solving in Arithmetic: A Review of the Literature." *Elementary School Journal*, 44: 396–403, (March 1944); 476–82, (April 1944).
12. Le Baron, W. A. "A Study of Teachers' Opinions in Methods of Teaching Arithmetic in the Elementary School." *Journal of Educational Research*, 43: 1–9; (September 1949).
13. McConnell, T. R. "Discovery Versus Authoritative Identification in the Learning of Children." (*University of Iowa Studies in Education*, Vol. IX, No. 5.) September 15, 1934. Pp. 13–62.
14. MacLatchy, J. H. "Number of Abilities of First Grade Children." *Childhood Education*, 11: 344–47; (May 1935).
15. Monroe, W. S., editor. *Encyclopedia of Educational Research*, Revised Edition. New York: Macmillan Company, 1950. See "Arithmetic" pp. 44–58.
16. Morton, R. L. "Estimating Quotient Figures When Dividing by Two Place Numbers." *Elementary School Journal*, 48: 141–48, (November 1947).
17. Mott, Sina M. "Number Concepts of Small Children." THE MATHEMATICS TEACHER, 38: 291–301, (November 1945).
18. Osburn, W. J. "Levels of Difficulty in Long Division." *Elementary School Journal*, 46: 441–47, (April 1946).
19. Ruch, G. M. "Relative Difficulty of the One Hundred Multiplication Facts with Special Reference to Textbook Construction." *Elementary School Journal*, 32: 369–77, (January 1932).
20. Stevens, M. P. "Teaching Arithmetic—Some Important Trends in Our Schools Today." *Grade Teacher*, 62: 54, (April 1945).
21. Thiele, C. L. *The Contribution of Generalization to the Learning of the Addition Facts.* (Contributions of Education, No. 763). New York: Bureau of Publications, Teachers College, Columbia University, 1938.
22. Washburne, C. W. and Vogel, M. "Are Any Number Combinations Inherently Difficult?" *Journal of Educational Research*, 17: 235–55, (April 1928).
23. Washburne, C. W. "When Should We Teach Arithmetic? A Committee of Seven Investigation." *Elementary School Journal*, 28: 659–65, (May 1928).
24. Wilburn, D. B. "Learning to Use a Ten in Subtraction." *Elementary School Journal*, 47: 461–66, (April 1947).
25. Wilson, G. M. "New Standards in Arithmetic." *Journal of Educational Research*, 22: 351–60, (December 1930).
26. Woody, C. "The Arithmetical Backgrounds of Young Children." *Journal of Educational Research*, 24: 188–201, (October 1931).
27. Wrightstone, J. W. *Appraisal of Newer Elementary School Practices.* New York: Bureau of Publications, Teachers College, Columbia University, 1938. Pp. 201–5.

ATTENDANCE AT THE TWELFTH N.C.T.M. CHRISTMAS MEETING

Oklahoma A. and M. College, Stillwater, Oklahoma
December 27-29, 1951

Alabama	1	Iowa	6	Ohio	2
Arkansas	18	Kansas	37	Oklahoma	132
California	1	Louisiana	5	Pennsylvania	2
Colorado	9	Michigan	3	Tennessee	8
Dist. of Col.	3	Missouri	4	Texas	29
Florida	1	Nebraska	1	Virginia	2
Illinois	6	New Jersey	2	Wisconsin	4
Indiana	4	New Mexico	4	TOTAL	284

Notice of Annual Business Meeting

As required by the By-laws of the National Council of Teachers of Mathematics, notice is hereby given to the members that the annual business meeting will be held Saturday, April 20, 1952 at the Hotel Fort Des Moines, Des Moines, Iowa.

M. H. AHRENDT, Executive Secretary

On Relationships of the Affiliated Groups with the National Council

JOHN R. MAYOR, *Chairman Committee on Affiliated Groups*
The University of Wisconsin, Madison, Wisconsin

THE AFFILIATED GROUPS AT THE 1952 ANNUAL MEETING

THE PLAN for participation of the Affiliated Groups, which was used at the Twenty-ninth Annual Meeting in Pittsburgh in 1951, will be followed for the Thirtieth Annual Meeting to be held at Hotel Fort Des Moines, Des Moines, Iowa, April 16–19, 1952, with one addition. This additional responsibility refers to the special sections sponsored by the Affiliated Groups in the immediate geographical area of Des Moines. At the 1952 meeting, sections will be sponsored by the Nebraska Council of Teachers of Mathematics and the Wisconsin Mathematics Council (see the February, 1952 number of THE MATHEMATICS TEACHER). While Affiliated Groups have sponsored sections at the Tenth and Eleventh Summer Meetings in Madison, Wisconsin, and Northfield, Minnesota, and the Twelfth Christmas Meeting in Stillwater, Oklahoma, the Des Moines Meeting will be the first Annual Meeting at which such sections have been featured. It should be observed that one topic to be discussed at the Delegate Assembly will be whether these special sections sponsored by individual Affiliated Groups should be continued.

The parts of the Des Moines program of special interest to the Affiliated Groups in addition to the sections sponsored by individual groups are:

Thursday, April 17, 8:00–10:00 A.M. First Session of the Third Delegate Assembly.

Friday, April 18, 10:00–11:30 A.M. Panel Discussion sponsored by the Committee on Affiliated Groups.

12:00 NOON. Luncheon for Official Delegates.

1:30–3:15 P.M. Second Session of the Delegate Assembly.

Questions which have been proposed for discussion at the Delegate Assembly were listed in the January number of THE MATHEMATICS TEACHER. The agenda for the Delegate Assembly, which will be determined from these questions and additional suggestions by Groups and members of the National Council, will be mailed to official delegates around April 1. A welcome to delegates will be given by Harry W. Charlesworth, president of the National Council. M. H. Ahrendt, executive secretary of the Council, will present a brief report on the Washington office. Others responsible for some of the important activities of the National Council will take part in the discussions at the Delegate Assembly. A special ceremony for presentation of Certificates of Affiliation to delegates of Groups which have completed affiliation since the Second Delegate Assembly will be held at the Thursday session of the Assembly.

The Delegate Assembly is planned to provide an opportunity for members of the National Council everywhere to have a more active part in the direction of existing services of the Council and in the initiation of new services. The Affiliated Groups should be sure to make full use of this privilege and responsibility. The Delegate Assembly can become an important agency in helping the National Council carry out its basic purpose to improve the teaching of mathematics at all levels of instruction.

Mary C. Rogers, who will preside at

the session sponsored by the Committee on Affiliated Groups, has planned the following program for this section.

Topic: *Specific Programs for Promoting Interest in Mathematics Education*
1—Among Students
2—Among Teachers of Mathematics
3—In the Community

Leader: MARY C. ROGERS, Roosevelt Junior High School, Westfield, New Jersey

Interest and Talent Development, WALTER H. CARNAHAN, Purdue University, Division of Education and Applied Psychology, Lafayette, Indiana

For Mathematics Class and Club: Magic Squares, BARNETT RICH, Richmond Hill High School, Richmond Hill, New York

The Mathematics Fair, H. W. CHARLESWORTH, President of the National Council of Teachers of Mathematics, East High School, Denver, Colorado

Introducing Mathematics to the Faculty, MARY A. POTTER, Supervisor of Mathematics, Racine, Wisconsin

The Mathematics Institute—An Aid to the Classroom Teacher, HOUSTON T. KARNES, Mathematics Department, Louisiana State University, Baton Rouge, Louisiana

The Conference on the Teaching of Mathematics, JOHN R. MAYOR, Department of Mathematics, The University of Wisconsin, Madison, Wisconsin

Official delegates of the Affiliated Groups and participants in the section sponsored by the Committee on Affiliated Groups will have lunch together following the Affiliated Groups section on Friday. This informal luncheon hour together will provide an additional opportunity for discussion of common problems.

REPORT OF NEW JERSEY CO-ORDINATING COMMITTEE

Early this year a copy of the Report of the Co-ordinating Committee of the Association of Mathematics Teachers of New Jersey was sent to the Committee on Affiliated Groups. Mary C. Rogers is chairman of the New Jersey Association's Co-ordinating Committee. Because this report was so comprehensive in coverage of problems of Affiliated Groups in their relationships to the National Council, the major part of the report is presented below. Affiliated Groups should find this New Jersey report of considerable help in planning their own programs.

Copies of annual reports of several of the Affiliated Groups were distributed at the Second Delegate Assembly and it is hoped that many more will be available for distribution in Des Moines. This Co-ordinating Committee report should be especially valuable to local officers responsible for the annual reports. Parts of Section 2 have been omitted here since a detailed report on the Speakers Bureau will appear in this section in an early number of THE MATHEMATICS TEACHER.

Report to the New Jersey Council

1. *New Jersey Membership in the National Council of Teachers of Mathematics.*
   - A. Membership in National Council should result from an appreciation of the intrinsic values of the services rendered by that organization.
   - B. The Association of Teachers of Mathematics of New Jersey should provide a reasonable amount of publicity toward that end.
     - a) At all meetings of our State Association
       1) Display poster or other publicity material furnished by National Council.
       2) Display sample publications, including *recent* issues of THE MATHEMATICS TEACHER, copies of Yearbooks, and various small publications.
       3) Give opportunity to our teachers of mathematics to examine this material, to place orders for copies of the publications, and to join the

Council. Classroom teachers, preferably not connected with the State Association or the National Council in any official capacity, but keenly interested in the services of these organizations, should assist as publicity agents in the display of materials and in the solicitation of membership.

b) Brief but pertinent publicity statements in the *New Jersey Mathematics Teacher.*

C. The Association of Mathematics Teachers of New Jersey should annually circularize the New Jersey teachers of mathematics for membership in National Council. Materials for circularizing should be furnished by National Council and postage paid by that Council.

2. *Speakers Bureau—New York Metropolitan Area.* (Under the direction of National Council.)

The possible services of a Speakers Bureau—the problems involved in its organization and administration—have been under official consideration by the NCTM officers and Board for more than a year. The more immediate study of problems involved, with recommendations for their solution, is now in the hands of an Advisory Committee, including a member from New York City, one from New York State, and one from Philadelphia, as well as members from New Jersey.

A preliminary report of this committee was made to the NCTM Board at the Summer convention at St. Olaf College in August, 1951. The Board at that time strongly favored trying out the Speakers Bureau service regionally before any attempt be made to extend the program. The Bureau was instructed to initiate trial service in the New York Metropolitan Area and evaluate its effectiveness. If this trial service is reasonably satisfactory, then similar regional services may be organized in other parts of the country. Eventually, there may be exchange of services between regions.

3. *Annual Affiliation Report—1951–1952.*

The Co-ordinating Committee expressed its commendation for the 1950–1951 Affiliation report and recommended the same procedure be followed for 1951–1952—a reappointment of the 1950–1951 committee; their report to be submitted to the New Jersey Council for approval or amendment.

4. *Appointment of State Representative to the National Council.*

This appointment has previously been made by the National Council. Recommendation was made at the Second Delegate Assembly that appointments henceforth be made by the local association. The Co-ordinating Committee approved of this action and recommended that the present incumbent be retained in this office.

5. *Appointment of Delegate and Alternate to the Third Delegate Assembly.*

Recommendation was made by the Committee that these appointments be made soon. The recommendation was accepted. Early action will be taken.

6. *National Council Convention Programs.*

Recommendations for improvement in future programs. The Co-ordinating Committee felt that there had been too much overlapping of program parts at recent conventions. It was felt that Continuity Groups and Study Groups carried over a period of days are more appropriate for Mathematics Institutes and Workshops and that more of such opportunities for study and exchange of ideas should be made available to teachers.

Furthermore, it was suggested that Discussion Groups should not be

(Continued on page 197)

Joint Summer Meeting of the National Council of Teachers of Mathematics and the New England Institute for Teachers of Mathematics

The Phillips Exeter Academy, Exeter, New Hampshire
August 21–24 and 21–28, 1952

THE TWELFTH summer meeting of the National Council of Teachers of Mathematics (August 21–24) and the fourth Institute for Teachers of Mathematics (August 21–28) sponsored by the Association of Teachers of Mathematics in New England will be held jointly next summer at The Phillips Exeter Academy. Lectures by outstanding speakers on timely topics in pure and applied mathematics are scheduled for each morning and evening. Three times a day study groups will meet in the academy's unusual classrooms. The material to be discussed will include methods of teaching, laboratory techniques, and trends and subjects of interest from the elementary level through the college freshman level.

MERRILL HALL AND LANGDELL HALL AT EXETER

Special features of the program are a study group on Group Dynamics, one on Effective Speaking in the Classroom (this will include opportunities to hear your recorded voice), and one on The Technique of Explanation. There will be a symposium on The Art of Teaching. The lecturers will include James R. Killian, Lillian Gilbreth, E. C. Berkeley, H. von Baravalle, Lillian R. Lieber, and Wayne Dancer. The complete program of lectures and study groups will appear in the April number of THE MATHEMATICS TEACHER.

There will be plenty of opportunities for "shop talk" and pleasant association with your companions at tea in "The Big Room" and at the Grill in the evening after the lecture. Come to Exeter for that stimulating atmosphere of congeniality, enthusiasm and hospitality for which the Council and Institute meetings are noted, and for that recreation obtained from the good fellowship of co-workers and the enriching inspiration of our mathematical leaders!

Phillips Exeter's beautiful and extensive campus is ideally located on a river just seven miles from the Atlantic Ocean and within 70 miles of the White Mountains. There is ample opportunity for ocean bathing, swimming, tennis, boating, golf, and mountain climbing. Visitors from other sections of the country will see in the charming and delightful town of Exeter a glimpse of historic New England at its best. Exeter is conveniently reached by automobile (5 miles from U. S. Route 1), by train (Boston

and Maine Railroad), by plane (to Manchester, N. H., or to Boston), or the Interstate bus from Boston will leave you right at the school.

The registration fee for the meetings will be $0.50 for members of The National Council and $1.50 for non-members. The registration fee for the period of The Institute is $2.00. The fee for room and board is $6.00 a day. Both single and double rooms are available in comfortable modern dormitories. The fees will cover all charges for the opening and closing banquets, the teas, tips, and the use of the tennis courts and swimming pool.

For further details concerning the meetings see the notice and registration blank in the April issue of THE MATHEMATICS TEACHER or write to Miss Ruth B. Eddy, 666 Angell Street, Providence 6, Rhode Island.

Immediate registration may be made by sending a deposit of $5.00 to Mr. H. Gray Funkhouser, Cilley Hall, Exeter, New Hampshire.

Affiliated Groups

(*Continued from page 195*)

scheduled simultaneously with Sectional or General meetings. Discussion Groups contribute much to National Council conventions, but they should run simultaneously with *each other.*

7. *Announcement of New Jersey meetings in National Council publications.*

The Committee recommended the appointment of a person to take the responsibility of placing such announcements regularly in THE MATHEMATICS TEACHER and in the Affiliated Groups *Newsletter.*

8. *Exchange of Teaching Aids—The Traveling Exhibit.*
9. *Exchange of Special Studies or Problems of Research.*

The Committee recognized the possible importance of these two matters but deferred any action or recommendation until a later date.

10. *Special Services to the Elementary Teacher.*

The Committee felt some action should be taken in this matter. It also recommended that action be taken to advance the good work already begun in New Jersey toward co-ordinating the interests of teachers in the Secondary Schools with the College program. It was felt that both of these are very important interest groups in New Jersey. The Committee recommended that committees be appointed to study these problems with an aim toward an early activation of services for these groups. Committee personnel should be carefully chosen and should be made up of persons enthusiastic about initiating functional programs; and also persistent and clever in seeing these programs through to a long-time successful finish.

11. *Institutes or Workshops sponsored by local or state associations.*

The Committee feels considerable interest in this problem. It has recommended that a committee be appointed to look into the possibility and feasibility of starting such an Institute or Workshop in New Jersey in the summer of 1952.

A Reminder

30th Annual Meeting at Des Moines

Have you made your reservations?

Will your school be represented?

NOTES ON THE HISTORY OF MATHEMATICS

Edited by VERA SANFORD
State Teachers College, Oneonta, New York

The Computus

THE VARIATION of Easter Day from March 22 to April 25 is accounted for in the rule established by the Council of Nicaea in 325 that "Easter Day is always the first Sunday after the Full Moon, which happens upon or next after the Twenty-first Day of March; and if the Full Moon happens on a Sunday, Easter Day is the Sunday after." Since much of the Church year depends on the date of Easter, this date had to be computed in advance. The problem was a complicated one, but it was a little simplified by a convention that substituted a lunar month computed by a time-honored method for the astronomical full moon. The computation of the date of Easter was given the name *Computus* or sometimes *Compotus* and this was the particular responsibility of the scholars in the monasteries. Several treatises bearing this name were written, one being the work of the Venerable Bede in the eighth century. The *Computus* of Anianus (1488) contains the following verse which will be recognized as the Latin form of a memoriter device:

"Junius aprils september et ipse nouember
"Dant triginta dies reliquis supraddit vnus
"De quorum numero februarius excipiatur."

To determine the date of Easter for a given year, several items are involved:
finding the age of the moon on March 21
finding the date of the next full moon
finding the day of the week on which this full moon falls
finding the date of the following Sunday.

The process was simplified by Carl Friedrich Gauss (1800) whose method appears in W. W. R. Ball's *Mathematical Recreations and Essays*, eighth edition, 1919, on which this account is based. In this method, constants m and n are determined from this table:

Years	1582–1699	1700–1799	1800–1899	1900–2099
m	22	23	23	24
n	2	3	4	5

Thus for 1952, $m=24$ and $n=5$.

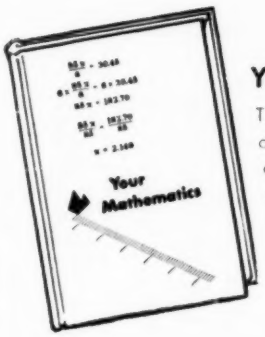
Then let a be the remainder when the number of the year is divided by 4
let b be the remainder when it is divided by 7
let c be the remainder when it is divided by 19
let d be the remainder when $19c+m$ is divided by 30
let e be the remainder when $2a+4b+6d+n$ is divided by 7.

Then the Easter full moon comes d days after March 21, and Easter Day is the $(22+d+e)$-th day of March or the $(d+e-9)$-th day of April. Thus for 1952, $m=24$, $n=5$, $a=0$, $b=6$, $c=14$, $d=20$, $e=2$. By substituting in the formula Easter Day is found to be the $20+2-9=13$ or the thirteenth day of April.

There are two exceptions to this. In 1954, $d=29$, $e=6$ and c is greater than 10. Easter Day would be April 25, but with this combination of values, Easter Day

(*Continued on page 204*)

FOCUSED on the Mathematical Needs of High-School Students



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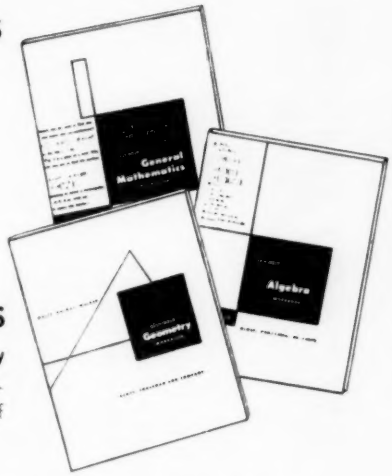

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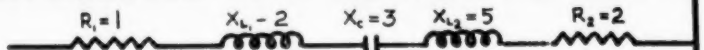
IMAGINARIES

HOW THEY ARE USED IN ELECTRICAL CIRCUITS

A Circuit offers impedance(Z) to AC alternating current } because of {

	SYMBOL
Resistance (R)	
Inductive reactance (X_L)	
Capacitative reactance (X_C)	

TO FIND IMPEDANCE OF A SERIES CIRCUIT...



1. Put a j before each X_L and a -j before each X_C

1 j2 -j3 j5 2

2. Add real and imaginary components separately.

$R_1 + R_2 = R_{TOTAL} = 1+2 = 3\ ohms$

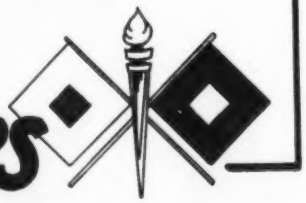
$X_{L_1} - X_C + X_{L_2} = X_{TOTAL} = 2-3+5 = 4\ ohms$

$Z = R_{TOTAL} + jX_{TOTAL} = 3 + j4\ ohms = Impedance$ (COMPLEX NUMBER)

3. The absolute value of the impedance is indicated by $|Z|$

$|Z| = \sqrt{R_{TOTAL}^2 + X_{TOTAL}^2} = \sqrt{3^2+4^2} = \sqrt{25} = 5\ ohms.$

MATH FOR THE SIGNAL CORPS



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IMAGINARIES

COMPLEX NUMBERS

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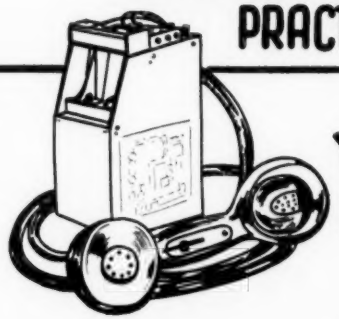

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FORMULA: $X_L = 2\pi fL$

$Z = R + jX_L$

$|Z| = \sqrt{R^2 + X_L^2}$

R = *Resistance of the coil* = 100 *ohms*

L = *Inductance of the coil* = 1 *henry*

f = *Frequency of the current* = 200 *cycles per second.*

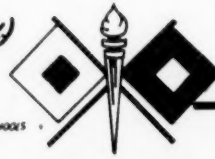
$X_L = 6.28 \times 200 \times 1 = 1256$ *ohms*

$Z = 100 + j1256$ *ohms*

$|Z| = \sqrt{10{,}000 + 1{,}587{,}536} =$ 1260 *ohms (approx.)*

MATH FOR THE SIGNAL CORPS

PREPARED BY SIGNAL SCHOOLS 6TH SERVICE COMMAND



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RESEARCH IN MATHEMATICS EDUCATION

Edited by JOHN J. KINSELLA
School of Education, New York University, New York, N. Y.

The Questions: To what extent is functional competence in mathematics improved through ninth grade courses in algebra and general mathematics? Which course is superior in this respect?

The Study: Beckmann, Milton W. *The Level of Mathematical Competency and Relative Gains in Competency of Pupils Enrolled in Algebra and General Mathematics.* Ph.D. dissertation. University of Nebraska. July 1951.

IN 1944 the National Council of Teachers of Mathematics assigned to the Commission on Post-War Plans the task of deciding the basic mathematics needed by all post-war citizens for personal use. The Commission emerged with a check list of twenty-nine competencies.[1] Possession of these came to be a commonly accepted definition of "functional competence" in mathematics. The authors of general mathematics textbooks for the ninth grade made it a point to call attention to the resemblance between their content and the twenty-nine competencies.

The principal purposes of Dr. Beckmann's study were "to measure the level of mathematical competency as defined by the twenty-nine competencies promulgated by the Commission on Post-War Plans, and the relative gains in such competencies as a result of the study of algebra and general mathematics in the ninth grade." (p. 6).

[1] R. Schorling, (Chairman). "The Second Report of the Commission on Post-War Plans." THE MATHEMATICS TEACHER. XXXVII. (May 1945), 195–221.

To accomplish these aims it was first necessary to construct a valid and reliable test of these competencies. An instrument of 110 items was devised and appraised by a jury of four specialists in mathematics education for readability and content validity. Preliminary tryouts involved fifteen university seniors and 155 high school students. As a result of these trials the test was improved and divided into two parts which could be administered during two class periods.

The test was then given in November and April to 1296 students in 42 Nebraska schools of varying size and geographical location. The fact that general mathematics or algebra, but not both, were required in the ninth grade was a crucial criterion in selecting the schools. Twenty-five of the schools, involving 734 students, constituted the General Mathematics Group and seventeen schools supplied 562 students for the Algebra Group. The California Mental Maturity Test was administered to all the students in February. The Algebra and General Mathematics Group means in this test were 108.97 and 106.65 respectively.

The reliabilities of the test computed by the split halves procedure and the Spearman-Brown formula ranged from .89 to .94. The difficulty of each of the 110 items was determined for the General Mathematics and Algebra Groups from the percentage of each group giving correct responses. Next, the gain or loss in difficulty of each item for each group of students was calculated. Then it was found that the General Mathematics Group's gains surpassed those of the

Algebra Group by statistically significant amounts at the one-per cent level for eleven of the twenty-nine competencies while the algebra group was superior in four cases. At the five per cent level of significance the respective number of items was one in each instance. Six of the remaining differences favored the Algebra Group and four the General Mathematics Group at levels which were not statistically significant.

The item validity of the test, based upon internal consistency, was determined by the use of the Pearson-r with the top and bottom twenty-seven per cent. The results revealed that only nine of the 110 items were probably invalid.

In the case of both groups the achievement was rather shocking. The Algebra Group moved from an initial mean of 46.54 to a final mean of 54.44; the corresponding statistics for the General Mathematics Group were 44.89 and 54.04. Only at the end of the year did either student group approximate a mean score of 50% correct. The annual gains in *items* correct amounted to 7.90 for the Algebra Group and 9.15 for the General Mathematics Group. Furthermore, at the end of the school year only 64% of the Algebra Group and 79% of the General Mathematics Group had half or more of the 110 items correct.

The question remained whether the 1.25 difference in gain in favor of the General Mathematics Group was statistically significant. An analysis of covariance was made to take into account the 2.32 mean score difference (108.97–106.65) in favor of the Algebra Group, in scholastic aptitude, as measured by the California test mentioned above. The results indicated that the 1.25 was large enough to be statistically significant beyond the one-per cent level.

Dr. Beckmann concluded with the observation: "If we accept the competencies as valid objectives for mathematics in the schools, then the low level of mathematical competency and the relatively small gains by ninth grade students enrolled in algebra and general mathematics in Nebraska, indicated by scores on this test of mathematical literacy, point out that one or more of the following conditions apply:

1. That our present curriculum in mathematics is poorly planned and developed.
2. That the teaching of mathematics is poorly done in our schools, or
3. That the competencies as proposed by the Post-War Commission have not been accepted as valid objectives by those planning the curriculum of the schools." (pp. 156–157)

This observer on the basis of this and other studies feels that it would be safe to assume that "functional competence" in mathematics needs further attention beyond grade nine. It should also be added that Nebraska holds no monopoly on the conditions revealed by Dr. Beckmann, as the study by Ohlsen showed with respect to the same competencies in another state.[2]

[2] M. Ohlsen, "Control of Fundamental Skills and Concepts by High School Student." THE MATHEMATICS TEACHER. XXXIX (December, 1946), 365–72.

History of Mathematics

(*Continued from page 198*)

will come a week earlier i.e., on April 18. In 1981, $d=29$, $e=6$, but Easter will come on April 19 instead of April 26 as would be given by the formula.

REFERENCES

1. *The Book of Common Prayer* of the Episcopal Church contains "Tables and Rules" for determination of the date of Easter and of the other movable feasts.

2. "Calendar." *Encyclopaedia Britannica*, (14th ed., 1947), IV, 571–72.

3. Ball, W. W. R., *Mathematical Recreations and Essays*, 8th edition, London: Macmillan, 1919, pp. 447–48.

4. Wilson, P. W., *The Romance of the Calendar*. New York, W. W. Norton, 1937.

Of these the *Romance of the Calendar* is excellent reading but it lacks documentation, which is unfortunate.

AIDS TO TEACHING

Edited by

HENRY W. SYER
School of Education
Boston University
Boston, Massachusetts

and

DONOVAN A. JOHNSON
College of Education
University of Minnesota
Minneapolis, Minnesota

BOOKLETS

B. 97—New Visual Education Techniques

Burgess Publishing Co., 426 South Sixth Street, Minneapolis, Minn.

Booklet; 79 pages; $8\frac{1}{2}'' \times 11''$; $1.25

Description: This booklet by Alfred Porter describes the use of felt marking-pens to produce a variety of teaching aids. After discussing the technique of using a felt-brush marking-pen, it illustrates its use in elementary and secondary classrooms. It suggests such uses as charts, drill cards, drawings, graphs, card games and sketches. It shows how the brush pen is a convenient device to label film, recording tapes, laboratory supplies, tools, books, athletic equipment, maps and shop drawings. An entire chapter of 29 pages is devoted to its use in the mathematics laboratory. Pictures and illustrations are used to suggest a great variety of ways to make abstract processes and symbols concrete.

Appraisal: The booklet is printed on good quality paper with a spiral binder and uses a great number of pictures to illustrate the items discussed. Not only will it suggest ways for the mathematics teacher to use brush pens but it will suggest a variety of ideas for making mathematics meaningful. Every teacher should have a brush pen and to get maximum use from it this book will be instructive.

B. 98—How Does It Work

School Service, Westinghouse Electric Corp., 306 4th Ave., Pittsburgh 30, Pa.

Booklet; 16 pages; free.

Description: This booklet uses the cartoon approach to present forcefully, briefly, yet interestingly such topics as atomic power, jet propulsion, electric lights, electric power, radio and television. For each topic, pictures or diagrams with concise captions illustrate the discovery of the device, how the device works, and its effects on society. It includes some mathematics such as Einstein's formula for mass and energy, graphs of temperatures and speeds of jet planes, and illustrations of the meanings of certain measurements. Each topic includes suggested projects for the reader to promote more complete understanding.

Appraisal: In view of the reading habits of present-day youth this style of book may be the best way to reach them. Obviously the treatment of each topic is superficial. However, attractive pictures and well-labeled diagrams make it possible for the reader to get some basic facts. The content of the booklet is primarily concerned with electrical principles rather than mathematics. It contains little advertising.

EQUIPMENT

E. 67—The Abacounter

The Abacounter Educational Company, 50 Broad Street, New York 4, N. Y.

An arithmetic teaching and learning aid; $39.50.

Description: The Abacounter is a dual-purpose device, designed by Dr. Nathan Lazar. It consists of a self-supporting birch wood frame, 38″×31″, divided into two equal rectangular frames. The frame at the left is a counting frame. Mounted in it are twelve horizontal removable aluminum rods, grouped five, five, and two, from top to bottom. On each rod are ten yellow bead counters, which are free to slide on the rod. The beads on each rod are in two groups of five beads each, the two groups being separated by a tubular plastic spacer, also free to slide.

The frame at the right is a seven-rod vertical abacus with several distinctive features. The vertical rods are removable, so that the number and color of the beads on any rod can be changed at will. There are twenty beads on each rod, grouped in fives with plastic spacers between groups. The beads of each rod have a distinctive color. The seven colors are chosen for vivid contrast. Each bead has a built-in spring which holds the bead at any position that it is placed on the rod, but which offers negligible resistance when the bead is moved up or down the rod.

General purpose: The purpose of the Abacounter is to afford a visual and tactual means for approaching basic number concepts, operations, and processes, so as to attach concrete meaning to them and thus to develop or strengthen arithmetical insight.

The Counting Frame: The counting frame has many specific uses, among which are the following:

1. To set up and understand one-to-one correspondence.
2. To count by ones, or by groups, up to 120.
3. To recognize small groups with little or no counting, (maximum of 5).
4. To identify numbers in terms of subgroups (Ex: 7=2+5, or 3+4, or 1+6) and to emphasize the usefulness of the groups, 5 and 10, in the quick identification of larger numbers.
5. To develop understanding of the relationship of addition to counting, and to see any addition fact as one of a related group, or pattern of facts.
6. To develop understanding of the relationship of subtraction to counting, and to see any subtraction fact as one of a related group, or pattern of facts, including corresponding addition facts.
7. To develop understanding of multiplication facts and their interrelationships and the relationship of multiplication to repeated addition.
8. To develop understanding of division facts and their interrelationships and the relationship of division to repeated subtraction.
9. To develop understanding of the distributive, associative, and commutative laws of arithmetic.
10. To develop understanding of fractional parts of groups, and the distinction between partition and measurement, or quotition division.

The Abacus: The abacus end of the Abacounter may be used for several purposes, among which are the following:

1. To develop or to clarify understanding of our place-value number system and its extension to include decimal fractions.
2. To develop understanding of the use of zero as a special place holder to indicate an "empty place."
3. To make clear the basic idea of conversion in the process steps (carrying and borrowing) and their relationship.
4. To gain an appreciation of a place-value system for performing addition, subtraction, multiplication, or division, by comparing the work of performing the same operation on the counting frame and on the abacus.

Appraisal: The Abacounter is durably and attractively constructed. Its over-all size is such that it is better adapted to teacher-pupil demonstrations than to individual pupil use. Its usefulness would be

increased appreciably if it were made in a smaller model, so that each pupil of a group of eight or ten could have one in his hands and could thus duplicate the demonstration immediately, under teacher guidance, or independently. The situation with only one large demonstration device in the classroom is analogous to that of a classroom having only one small blackboard and no paper and pencils for the children. The learning of arithmetic by a class of thirty pupils would be rather slow under these conditions.

As a demonstration device for the purposes already listed, the Abacounter is a most effective teaching and learning aid.

The use of an abacus, of course, does not get away from symbolism entirely. The substitution of one bead for ten is just about as symbolic as the idea of a numeral with positional value. However, this fact does not lessen the importance of the abacus as an effective device for a transitional stage of learning. The dual nature of the Abacounter makes it possible to emphasize the transitional stage most effectively.

The present price of $39.50 net for the Abacounter will seem high to small schools operating under an extremely limited budget. They will hope for a time when a more inexpensive model will be available. Meanwhile, they will continue to use cruder, home-made, or pupil-made devices, to accomplish the same purposes, though somewhat less effectively. (Reviewed by Joseph H. Randall, Principal of Franklin School, West Newton, Mass.)

E. 68—Arith-O-Cards (Addition)

Arith-O-Card Company, 157 South Mentor Avenue, Pasadena 5, Cal.

Card game; 119 cards, standard playing card size, plus 4 addition tables and a set of directions; in box; $1.00 plus postage.

Description: Cards are made up of all combinations for adding two one-digit numbers as well as the answer cards. Thus cards include 0, 0+0, 1, 1+0, 0+1, 2, 2+0, 0+2, 1+1, · · · through 9+9, 18, a total of 119 cards.

The object of the game is to make as many books as possible and to go out first, similar to the game of Rummy. A book consists of two or more cards of equal value. For example: 3+2 could be matched with 2+3, 4+1, 1+4, 5+0, 0+5 or the answer card 5.) Any time a player has a card which goes with any book, whether his own, or someone else's, he may lay the card down for his own credit. Other variations of the rules are also suggested.

Appraisal: This game is useful in drilling on the addition combinations and it does it in a manner that is enjoyable to youngsters. Since only one-digit numbers are used in the combinations, many possible cards are omitted. Thus 16 has only 9+7, 7+9, 8+8, and the answer card 16, whereas it could have had 10+6, 6+10, 11+5, 5+11, and so on. However, if all such possibilities had been included, the deck would have been much too large and unwieldy. One interesting feature includes on the same card two ways of indicating a sum. In diametrically opposite corners appear 3+2 and $\begin{array}{r} 3 \\ +2. \end{array}$

Because of the inclusion of the card 2+3 the student also learns that 3+2 has the same answer as 2+3.

In games of this type it must be remembered that they represent good forms for drill, but basic understandings must still be developed by the teacher. (Reviewed by William H. Glen, Jr., Pasadena, Cal.)

E. 69—Arith-O-Cards (Multiplication)

Arith-O-Card Company, 157 South Mentor Avenue, Pasadena 5, Cal.

Card game; 137 cards, standard playing card size, plus 4 cards with a multiplication table and a set of directions; $1.00 plus postage.

Description: Cards are made up of all combinations for multiplying two one-digit numbers as well as the answer cards.

Thus cards include 0, 1×0, 0×1, 2×0, 0×2, · · · through 9×9, 81, a total of 137 cards. Two to six players are preferable and they may or may not play as partners.

The object of the game is to make as many books as possible and to go out similar to the game of Rummy. A book consists of two or more cards of equal value. (For example: 3×4 could be matched with 4×3, 2×6, 6×2, or the answer card 12.) Any time a player has a card which goes with any book, whether his own, or someone else's, he may lay the card down for his own credit.

Appraisal: This game is useful in drilling on the fundamental combinations of multiplication, and it does it in a manner that is enjoyable to youngsters. Since only one-digit numbers are used as factors, many combinations are omitted. Thus 64 has only two cards in the deck, 8×8 and the answer card 64, when it could have had many others of course. However, if all possible factored combinations had been retained the deck would have become extremely large and unwieldy. Besides giving drill in the basic combinations of multiplication, the cards are printed so that when held in the hand it is equally likely that

$$3\times5 \text{ or } \underset{\times 5}{3}$$

will appear since these are printed in diametrically opposite corners. The student then recognizes that there is more than one way to indicate a product. Also the student learns that 3×5 has the same answer as 5×3.

Four cards with a multiplication table come with each deck so that if a child needs to find answers, he may do so by having the card in his hand and thus avoid any embarrassment.

The game can be adapted to the level of a group also. If a group only knows combinations through the 4's then only those cards need to be used. As they begin to learn new combinations, the corresponding cards can be returned to the deck.

It must be remembered that the game is just another good approach to drill. The child would not learn from this when to use the proper arithmetic operations. Basic understandings must still be developed by the teacher. (Reviewed by William H. Glen, Jr., Pasadena, Cal.)

E. 70—Fraction Trainers

Robinson Howell Company, 641 Mission Street, San Francisco 5, Cal.

Models; Plastic Circles; $3.75.

Description: This device consists of a black plastic circular board with mounted pegs upon which can be fitted the following equal size circles: one red plastic whole circle to represent unity; two red plastic semi-circles; three yellow plastic third pieces of another circle; four yellow plastic fourth pieces of another circle; six blue plastic sixth pieces of another circle; and eight blue plastic eighth pieces of another circle. Thus six layers of circles can be fitted or removed from the fraction trainer. A pamphlet accompanies this instrument and offers many valuable suggestions as to meanings and understandings in fractions which a pupil can discover for himself.

Appraisal: A teacher or individual pupil may manipulate these fractional parts of a circle to show the various relationships between fractions of different denominators. Most pupils will enjoy handling this device because of its brightly colored pieces and also because of the unique way in which each layer of fractions must be assembled over the pegs onto the fraction trainer. Being concrete, and not abstract, these pieces are able to help students visualize relative sizes, make comparisons, identify, and manipulate fractions. It is suggested that only one layer of fractions at a time be used on the fraction trainer. This device could be used to the greatest advantage in the fifth grade before formal work in adding or subtracting like fractions is studied. It has been found helpful and stimulating in the reviewer's sixth grade. (Reviewed by George Janicki, Elm School, Elmwood Park, Ill.)

E. 71—Math-o-Block

Number blocks; 23 wooden blocks; $3.95, teacher's set, $10.95.

D. T. Davis Co., 178 Walnut Street, Lexington, Kentucky.

Description: These number blocks are designed to teach the beginning fundamentals of arithmetic in a concrete manner. Each block has one of the digits of our number system painted on its faces. There are nine 1-blocks, four 2-blocks, three 3-blocks, two 4-blocks and one block for each of the numbers 5, 6, 7, 8 and 9. The thickness of each block is proportional to the number it represents and thus, can be used to show the true value of each digit of our number system except zero. The different numbered blocks are each painted a different color to aid in distinguishing numbers. The blocks can be used to show the number combinations up to nine. For example, the 6-block can be shown to be equal to six 1-blocks, or three 2-blocks or two 3-blocks or a 4-block and a 2-block. Similarly they can be used to show simple subtraction, multiplication and division.

Appraisal: These attractive wooden blocks will furnish the primary teacher with a simple device to build an understanding of numbers and of the fundamental operations. It is unfortunate that the set is as expensive as it is, but the effect they should have on the learning of basic concepts should make them well worth the price.

FILMS

F. 68—Origin of Mathematics

United World Films, Inc., 1445 Park Avenue, New York 29, N. Y.

Educational Collaborator: Samuel T. Alexander

B&W ($45); 1 reel; 11 min.

Description: The film opens with a cave man writing on the cave wall. He is using animals for numbers. The Egyptian symbols are then shown with the remark that multiplication would be very difficult in this system. The Babylonian number system is illustrated. It is explained that they counted in sixes and sixties. Three hundred sixty was used as the common denominator for all fractions. Illustrations of how this method of counting has been carried over to the present time are given such as: the clock with sixty seconds and sixty minutes; the yardstick with thirty six inches; the circle with three hundred sixty degrees; and the globe with three hundred sixty degrees in longitude.

The theorem of Pythagoras is explained and then the Roman number system is shown. The fact that multiplication would be hard in the Roman system is depicted.

The counting board or abacus is used for adding and multiplication. Examples of this are shown. Different types of the abacus are also illustrated.

The film concludes with various scenes showing machine inventions because of mathematics, and how bridges, ships, multiple presses turning out books, chemists, architects, new tools, and the wind tunnel all use mathematics for precision measurement.

Appraisal: For the evaluation of this film there were twelve teachers of mathematics present. They considered this film in its entirety as good for introducing new material, to augment explanations, to provide a common experience and to motivate. 78% of the teachers thought the film was either fair or poor in connection with developing skills. It was agreed that the film could be used from the 7th grade to the college level. Opinion on this was about evenly divided with 7% favoring the 12th grade and 23% favoring the 8th grade. Between ten and sixteen per cent of the teachers favored the other grades. 57% of the teachers thought the film was suited to courses in arithmetic and algebra, and 18% listed geometry as the course best suited. 25% of the teachers thought the speed of development of ideas and duration of scenes was too fast. 30% of the teachers found inaccuracies in the film.

(Continued on page 214)

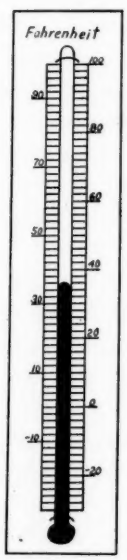
APPLICATIONS

Edited by SHELDON S. MYERS
Department of Education, Ohio State University, Columbus, Ohio

ONE OF the reasons why some teachers are reluctant to employ applications to any extent is because they believe that practice always follows theory and is separated from it. Thus they will devote sufficient time for the development of a process in mathematics and then, in the limited time (if any) available, will provide the students with problems, some of which might be called applications in a practical sense. It is our purpose in this issue of THE MATHEMATICS TEACHER to show how applications can be used in developing the fundamental processes of addition and subtraction with directed numbers, thus providing an integration of theory and practice with the concomitant effects of increased motivation and better understanding. This issue was written in collaboration with Oscar Schaaf of the Ohio State University School, who employed these methods as part of a doctoral study on the development of generalization ability in beginning algebra under the direction of Dr. Harold P. Fawcett. We wish to emphasize the significance of the *approach*, rather than claim originality for the specific applications. Furthermore, it should be apparent that certain words and ideas employed in the following material have had adequate prior development which cannot be described here because of limitations of space.

Al. 12 Gr. 9 Signed Numbers and the Thermometer

The following series of problems might be given at the very beginning of work with signed numbers before any operating rules have been developed.



1. What is the reading on the thermometer?
2. Suppose the thermometer read $-23°$. What would be the meaning of the reading? What does the reading in question one mean?
3. Locate and label the following readings on the thermometer scaled at the left of this sheet: (A) 46° (B) $-3°$ (C) 63° (D) $-8\frac{1}{2}°$ etc.
4. What is the reading halfway between the following readings? (A) 74° and 110° (B) 0° and 10° (C) $-8°$ and 8° (D) 4° and $-2°$ (E) $-3°$ and $-9°$ (F) 6° and $-8°$
5. Which number is farther from 0°? (A) 6° or $-10°$ (B) 26° or 18°
6. Which number is farther from 12°? (A) 18° or 8° (B) 26° or $-5°$
7. Which number is farther from $-2°$? (A) 7° or $-10°$ (B) 2° or $-8°$
8. If the thermometer starts at 40° and moves to 47°, how far has it moved? What was the direction of the change, up or down?
9. If the thermometer starts at 40° and moves to 33°, how far has it moved? What was the direction of change?
10. If the thermometer stands at 10° and then drops 16°, what will the new reading be?

11. If the thermometer stands at $-12°$ and rises 5°, what will the new reading be?
12. If the thermometer stands at 23° and falls 27°, what will the new reading be?
13. If the thermometer moves from 14° to $-10°$, how far and in what direction (up or down) did it move?
14. If the temperature at the present time is $-10°$ and we are told that the temperature has dropped steadily at the rate of $\frac{1}{2}°$ every hour, was the temperature higher or lower six hours *ago* and by how much?
15. At 8 A.M. the temperature was 40°. If the temperature rises at the rate of 2° every hour, what is the temperature at 2 P.M.?

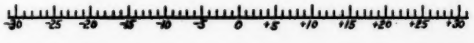
Al. 13 Gr. 9 Situations Requiring Addition of Signed Numbers

Here are a series of problems which might be used a day or two after the foregoing set. It should be noted here that the foregoing problems illustrate miscellaneous operations with signed numbers, while the following set is selected to involve only the addition process.

1. At various times during the first quarter of a game against Michigan the Buckeyes made the following four series of downs. Find the net gain or loss of yardage on each series of downs. (A) 5 yds. gain, 3 yds. gain, 10 yds. gain, 7 yds. gain, 4 yds. gain (B) 2 yds. loss, 4 yds. loss, 3 yds. loss, 1 yd. loss, 5 yds. loss (C) 10 yds. gain, 5 yds. loss, 7 yds. gain, 3 yds. loss, 12 yds. gain (D) 6 yds. loss, 7 yds. loss, 9 yds. gain, 3 yds. loss, 2 yds. gain
2. In forecasting weather the hourly change in temperature is often more helpful than the actual temperature. What is the net change in temperature if the hourly change has been reported to be as follows: (A) up 2°, up 5° up 1°, up 4° (B) down 1°, down 3°, down 2°, down 6° (C) down 2°, down 1°, up 2°, up 3° (D) up 3°, up 2°, down 1°, down 4°
3. What is the total change in value for the past week of one share of A.T. and T. stock if the daily changes are as follows: (A) down 1, down $\frac{1}{2}$, down $\frac{1}{8}$, down $\frac{3}{8}$, down $\frac{1}{8}$, down $\frac{1}{4}$ (B) up $\frac{1}{8}$, up $\frac{3}{8}$, up 1, up $1\frac{1}{8}$, up $\frac{1}{2}$, up $\frac{5}{8}$ (C) down 1, down $1\frac{1}{2}$, up 2, down $\frac{1}{4}$, up $\frac{3}{8}$, up $\frac{1}{2}$ (D) up 1, down $\frac{1}{8}$, down $\frac{1}{2}$, up $\frac{1}{4}$, down 2, down 1
4. In the middle of the 15th Century, the European merchants developed the practice of marking barrels and boxes with a positive sign (+) if they were over regulation weight and with a negative sign (−) if they were under. Thus +7 on a barrel meant 7 pounds overweight and −5 meant 5 pounds under regulation weight. If a merchant bought the following sets of barrels and boxes with the indicated markings, find the total amount that each set was over or under weight. (A) +15, +4, +6, +3 (B) −20, −5, +3, −2 (C) −10, +6, +9, +3 (D) −5, −4, +7, −6, (E) −3, −6, −10, −7 (F) $-3\frac{1}{2}$, $+4\frac{2}{3}$, $-6\frac{4}{5}$

Al. 14 Gr. 9 Use of the Directed Number Scale as Part of a Planned Series of Experiences with Signed Numbers

At this stage students are ready for a more formalized and a somewhat more abstract experience with signed numbers. The term "directed number" can be introduced in connection with the number scale below. It must be noted that the students have already been informally brought into contact with the meaning of "directed" in the foregoing problems.



Add the following signed numbers by making use of the scale above:

1. $+6$ / -2 Students can be introduced to the use of the scale with this problem by linking it to one of the previous examples. For instance, the students can imagine the scale to be a football field with the line of scrimmage at the zero point and the team involved

advancing toward the right so that gains can be recorded by increasing numbers. Using the above problem, the first play is a gain of 6 yards which is recorded by counting off 6 units to the right. *From this point,* the next play, which is a loss of 2 yards, is recorded by counting off 2 units to the left, ending up at +4. This analogy can be used to develop the procedures of adding numbers on the number scale and can be discarded as soon as these procedures are familiar. Other problems to be worked are as follows:

2. $+7 + (-4)$
3. $-3 + (-6)$
4. -3 / -5 — 5. -6 / $+10$
6. 0 / $+5$ — 7. $+8$ / -8
8. $(-8)+(-6)$
9. $(0)+(-9)$
10. $+15$ / -4 / -6 / $+3$ — 11. -8 / -1 / $+5$ / $+6$ / 0 / $+12$

12. Make up a story problem that will fit number 11.
13. Write a rule which will tell you how to work problems 1–11 without making use of the scale.

Question 13 is the culmination of all of the experiences described so far in this discussion. These experiences stress the inductive, generalizing type of learning, rather than the formal, deductive type. At this point, Mr. Schaaf's beginning algebra class worked on stating a satisfactory rule and finally agreed as a group on the following four-part rule which represents their own wording:

1. Add arithmetically the absolute values of all the positive numbers.
2. Add arithmetically the absolute values of all the negative numbers.
3. Subtract arithmetically the sum with the smaller absolute value from the sum with the larger absolute value.
4. The sign of the final answer will be the same as the sign of the sum with the larger absolute value.

Such a rule, difficult as it may sound, when made by students as an outgrowth of their own experiences and judgments, is far more meaningful than one which is arbitrarily memorized and then applied. In the latter case, memorization does not assure ability to apply. In the former case, it is striking that the rule has already been extensively employed before it is formally stated, and little effort is needed by students at this stage to reproduce it or apply it.

At this point some exercises similar to the previous set may be given in order to provide some practice in using the rule deductively. The over-all time is probably less than in the formal theory-to-practice approach and meanings are probably more firmly entrenched. Some experimental data on this point may be available in the future.

After using the above rule for a while, one of Mr. Schaaf's students became dissatisfied with it and proposed the following amended version:

1. same as above
2. same as above
3. Subtract arithmetically the smaller sum from the larger.
4. If the sum of the absolute values of the positive numbers is larger, the final answer will be positive, if it is smaller, the final answer will be negative.

Attempts by students to refine their statements in order to describe a situation more logically are to be encouraged. Furthermore, just as in Nature Proof students are led to see that geometric assumptions are man-made, so in algebra students can be led to see that mathematical rules are man-made agreements.

The development of a rule for subtraction of signed numbers is similar to that

for addition. A series of graduated practical experiences with the subtraction process involving signed numbers is provided leading to the performance of the operation on the directed number scale and finally culminating in a formulation of a rule by the students. Following are two examples of practical experiences.

Al. 15 Gr. 9 Situations Requiring Subtraction of Signed Numbers

The new temperature reading is in the first column, while the present temperature reading is in the second column. Find the difference in temperature and indicate whether the direction of change is up or down.

	New Temperature	Present Temperature	Difference	Direction
a.	+55°	+45°		
b.	−10°	−15°		
c.	+16°	+23°		
d.	−13°	− 7°		
e.	+ 6°	0°		
f.	+12°	− 5°		
g.	0°	+11°		
h.	− 3°	+10°		
i.	−14°	0°		
j.	+ 6°	− 9°		
k.	0°	−12°		
l.	−10°	+ 2°		

Some hay farmers make it a practice to fasten a tab on the wire of a bale of hay indicating the amount it weighs over or under one hundred pounds. Thus a +7 on a tab means 7 pounds over one hundred pounds, while a −5 means 5 pounds under one hundred pounds. The first column indicates the tab readings for your friend's hay bales, while the second column indicates the tab readings for your bales. In each case find out how much your friend's bale is more or less than yours.

When introducing the use of the directed number scale for subtracting signed numbers, the previous illustration involving temperature change could be employed as follows:

Subtract: −4
<u>−8</u>

The −8 can be considered to be the "present temperature," while the −4 can be considered to be the "new temperature." On the number scale −8 is first located by the student who then counts to −4. The *amount* of the count is the numerical difference, while the *direction* of count indicates the sign of the difference—positive for right and negative for left. This latter convention should be discovered by the students from the known answer of the example as a temperature problem. After working a number of subtraction problems on the directed number scale, Mr. Schaaf's class made the following three-part rule for subtraction:

	Friend's Bale	Your Bale	Difference	More or Less Than Yours
a.	+ 8	+ 5		
b.	+ 6	+ 9		
c.	−10	0		
d.	+ 5	− 2		
e.	−10	− 5		
f.	− 1	− 6		
g.	0	−10		
h.	− 7	+10		
i.	+ 2	− 5		
j.	− 8	+ 3		
k.	− 7	− 3		
l.	− 2	− 9		

1. If you have a subtraction problem with both negative and positive numbers in it, you add arithmetically the absolute values of the numbers and then follow number 3 below.
2. If you have two numbers with the same sign, subtract arithmetically the smaller absolute value from the larger and follow number 3 below.
3. If the number being subtracted is larger (in signed value, not in absolute value) the answer will be negative. If the number being subtracted is smaller, the answer will be positive.

The above rule is a rather unusual statement for ninth graders to make. This way of stating the subtraction rule does not appear in textbooks, and yet it constitutes a meaningful and mathematically sound statement for these freshmen.

Closing note: The authors are aware of the various operational interpretations of the subtraction algorism involving positive numbers. It becomes an interesting problem to determine what operational interpretations apply to signed numbers and what practical applications illustrate each of the latter interpretations. The editor of this department will welcome correspondence on this matter.

Aids to Teaching

(*Continued from page 209*)

The teaching methods in the film were partially conducive to learning for 59% of the evaluators. Two of the evaluators found the photography was fair while seven found that it was good. The sound was good for five of the teachers, but three found it was fair and two said it was poor. Six of the evaluators said that the content of the film could be just as effectively and efficiently presented in some other way, but five said that it could not. Two-thirds of the teachers said they would use this film in their classroom. (Reviewed by A. DiLuna, R. Fleet, K. Hathaway)

F. 69—Arithmetic for Beginners, Parts I, II, III

Bailey Film, Inc., 2044 North Berendo, Hollywood 27, Calif.

Director, Constance E. Amsden
B & W ($90); 3 reels.

Description: These films use toy animals to teach the simple addition and subtraction facts. Part I presents the addition facts $1+1=2$, and $2+1=3$. Part II presents $1+3=4$, $2+2=4$, $4+1=5$, and $2+3=5$. Part III presents $1+5=6$; $2+4=6$, and $3+3=6$. Each additional fact is presented in both orders, for example, $1+3=4$, and $3+1=4$. Each addition fact is related to its corresponding subtracting fact, for example, $2+3=5$, or $5-2=3$ and $5-3=2$. These facts are developed by bringing the toy animals together in the picture to illustrate addition, and having the animals leave the scene to illustrate subtraction. As the facts are illustrated by toy animals, the examples are written in symbols at the same time the facts are described by the commentator.

Appraisal: These films have very good photography and slow, clear commentary adapted to primary pupils. They do a good job of relating addition facts to the related subtraction facts. They should make the operations of addition and subtraction have concrete meaning. However, it seems strange that each example is flashed on the screen in symbols before the concrete illustration of the example is shown. The best use of this film will probably be for the training of teachers, since the primary teacher should use concrete objects herself in the same way as used in the film to build an understanding of addition and subtraction facts.

DEVICES FOR A MATHEMATICS LABORATORY

Edited by EMIL J. BERGER
Monroe High School, St. Paul, Minnesota

Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. Or if that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

DEVICE FOR LOCATING IRRATIONAL NUMBERS ON A NUMBER SCALE

In the usual development of the structure of the real number system in intermediate or second course algebra a rather unhappy realization presents itself for the student when irrational numbers are introduced. This may be true because rational numbers seem to have such a natural relationship to the positive and negative integers—i.e., they can be defined entirely in terms of them, while irrational numbers can only be adequately defined by following a line of reasoning that is usually beyond the comprehension of high school students. It is not uncommon, therefore, for irrational numbers to be introduced in terms of what they are not—e.g., "numbers which cannot be expressed as the quotient of two integers." The experience of the department editor has been that such a definition is far from satisfactory. How then is the teacher to present the concept of irrational numbers? With a device? Perhaps!

The device suggested in this article for giving meaning to irrational numbers and relating them to rational numbers is so simple that it is really not a device at all, but a number of teachers who have seen it have smiled appreciatingly at its simplicity and the classmates of the boy who produced it have endorsed it with grins of satisfaction. Hence it is included here for the reader's consideration. To illustrate how naturally it may be introduced a little background information is presented first.

In the usual scheme of defining rational numbers the process of division with integers is first defined. Since an expression for the quotient leads naturally to numbers like c/d, it is necessary to recognize a new set of numbers of the form c/d, where c and d are any positive or negative integers, except that $d \neq 0$. But a ratio is an indicated division, so all numbers of the form c/d are called rational numbers. Since it is possible to divide a line segment into any number of equal parts, all rational numbers may be represented on the number scale in terms of a unit segment or the length of the line from 0 to ± 1. This provides a clue for giving meaning to the irrational numbers. Irrational numbers may also be represented on the number scale. Thus rational and irrational numbers may be related by showing that both of them can be located on the number scale. But let us proceed with the method of locating irrational numbers.

Procure a piece of welding rod 36″ long, a rubber suction cup with a bolt molded into the top, and a metal cap from a cooking extract bottle. Rubber suction cups of the type described are available commercially. Mark the welding rod at a point 4″ from one end and bend it into a right angle at this point. If the device is to give accurate results, care must be taken to insure that the angle is as nearly 90° as possible. Drill three $\frac{1}{8}$″ holes in the metal cap—one through the center of the top, and two through the lip at opposite ends of a diameter. (See Fig. 1.) To assemble the device slip the cap over the bolt on the

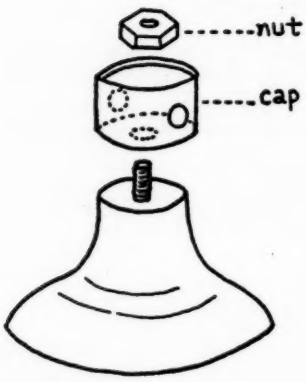


FIG. 1.

suction cup and turn the nut down, but do not turn it down tight because the cap must be able to rotate freely. Slide the long arm of the rod through the holes in the lip of the cap as shown in Figure 2. The rod must slide freely; if it does not, enlarge the holes slightly.

The device is now ready to be used. Construct a number scale on the blackboard in the following manner: Draw a horizontal line 72″ long, divide it into 4″ segments, and locate the numbers 0, ± 1, ± 2, etc. at the proper division markers. Compress the suction cup over the zero point and apply enough pressure to make it adhere to the blackboard. Next slide the long arm of the rod through the cap so that the right angle vertex C will coincide with the number $+1$. This is the position of the rod shown in Figure 2. If the device has been constructed carefully $AC=CB=1$. Since $(AC)^2+(CB)^2=(AB)^2$, it follows that $AB=\sqrt{2}$. Swing the rod down so that B will coincide with the number scale at C'; then $AC'=+\sqrt{2}$. In this position the right angle vertex will be at F' and the end of the rod H will be somewhere on the dashed line $E'F'$ between A and E'.

Rotate the rod back to its first position and slide the long arm to the right so that C will coincide with C'. Now B will be at B' and it can be seen that $(AC')^2+(C'B')^2=(AB')^2$. But $AC'=\sqrt{2}$, and $C'B'=BC=1$ as before. Therefore $(AB')^2=3$, and $AB'=\sqrt{3}$. Swing the rod down so that the top of the rod coincides with C''; then $AC''=+\sqrt{3}$. In this position the right angle vertex is at F'', and the end of the rod H will lie somewhere along the dashed line $E''F''$ between A and E''.

By a repetition of the above process the numbers $+\sqrt{4}$, $+\sqrt{5}$, etc. may be located on the number scale in rapid succession. The thing which brings the smiles to students' faces is that $+\sqrt{4}$ and $+\sqrt{9}$ coincide with $+2$ and $+3$.

The numbers $-\sqrt{2}$, $-\sqrt{3}$, etc. may be

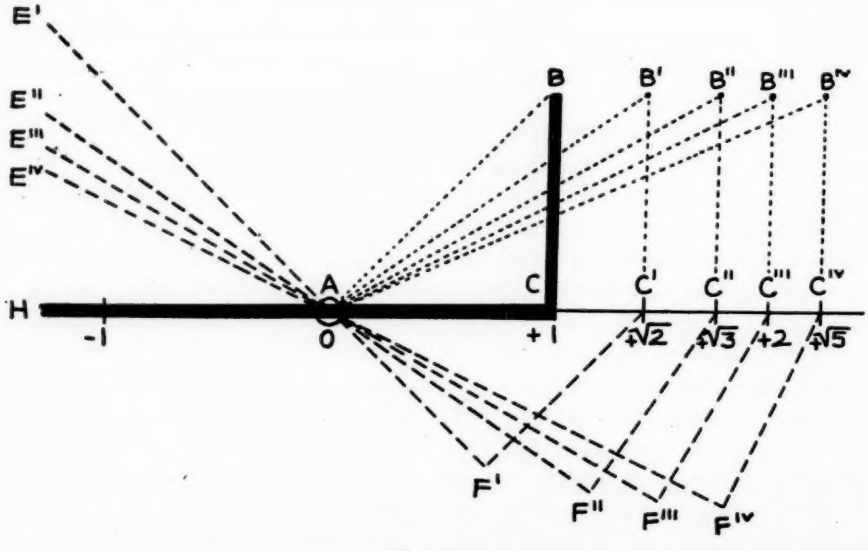


FIG. 2

located by pulling the rod out of the cap holder and sliding it through in the opposite direction so that the arm BC will be above the number scale on the left-hand side of the number 0.

THE MATHEMATICS LABORATORY
Monroe High School

MAKING A SLIDE RULE WITH METER STICKS

Perhaps the simplest way of describing the ordinary slide rule is to say that it is a device for performing computations with logarithms mechanically. The implication for teaching in this statement is that the relation between logarithms and the slide rule must be made as meaningful as possible. This can be done by illustrating concretely how logarithmic scales and linear scales are related, and how operations with logarithms may be effected by manipulating linear scales.

A very convenient linear scale to use for this purpose is the ordinary meter stick. Its division into 1,000 millimeters suggests an easy method of locating the decimal values of numbers from 0 to 1 correct to the nearest one-thousandth. Thus it is possible to locate the logarithms of all numbers from 0 to 10 correct to three decimal places; e.g., log 2 (=0.301) is located at 301 millimeters, log 3 (=0.477) at 477 millimeters, etc. To do this cut out small paper tabs with an arrow at one end, write a number and its logarithm on each tab, and paste them on the meter stick so that the arrows point to the proper millimeter markings. Figure 3 illustrates how this is done for the number 3 and its logarithm (0.477).

Two meter sticks should be prepared in the manner described. The logarithms of the integers from 1 to 10 should be located first. Following this the logarithms of such numbers as 0.5, 1.5, etc. may be located if desired. One thing which should be pointed out to the student is that the number at each point is the antilogarithm of the logarithm located on the millimeter scale. By using the two scales suggested it is possible to illustrate concretely that addition and subtraction of logarithms can be accomplished geometrically by adding or subtracting line values of logarithms. This is the same as using the "C" and "D" scales on an ordinary slide rule. It is only one more step to show that the meter sticks can be used to multiply and divide numbers.

FIG. 3

The method described can be extended to include the operations of squaring a number and finding its square root. To prepare the "A" scale needed for this work it is suggested that the number of millimeters denoting any logarithm be doubled and every millimeter division on the meter stick be used to represent two-thousandths.

Invariably when the above method is used some student who is simply itching to start using his slide rule comes up with this question: "How are the decimal values of logarithms for the numbers 1 to 10 located on the short scales which are on the slide rule?" If the question does not come up the teacher may well ask it himself. The answer of course is simple. The short linear scale on the slide rule can be divided into 1,000 parts in the same way that the meter stick is divided into 1,000 parts, and the logarithms located in the same manner on the short scale. If the student voices the opinion that 1,000 division marks on a short scale will be hard to distinguish he may be told that the short scale can be divided into 100 parts and the logarithms located correct to the nearest one-hundredth.

THE MATHEMATICS LABORATORY
Monroe High School

MATHEMATICAL RECREATIONS

Edited by AARON BAKST
135-12 77th Avenue, Flushing 67, N. Y.

THIS IS an age of *preparedness* in which we live. We are being warned of imminent dangers that may beset us in this troubled world of ours. We are threatened with all sorts of calamities. There is the danger of atomic explosions. Chemical warfare may also hit mankind. There is the danger of bacteriological warfare. All these are dangers which we must face and prepare ourselves to meet if we are to survive. We are also told that mathematics must play a very important part in our preparing ourselves. We are told that mathematics is basic to our efforts to defend ourselves. We are also told that mathematics is responsible for most of, if not all, the discoveries which have led to the development of the instruments of warfare which stagger our imagination. And in the same breath we are told that mathematics is and will be responsible for our survival. Mathematics is both the villain and the benefactor. But mathematics may also be hit very hard. . . . It can suffer destruction if we are not prepared, mathematically speaking.

Let us imagine that by some unexpected cataclysm all the trigonometric tables were suddenly destroyed. Everything else in mathematics is left intact, but trigonometric tables are wiped out so completely that not even a single numerical value of a trigonometric function was left on the face of the earth. This would probably call for a celebration in many a mathematics classroom. No more measuring the height of a tree or the distance across the river. Field work in mathematics would be thrown out of a job. Transits, sextants, angle mirrors, alidades, Jacob staff's, and all other paraphernalia would be just museum curiosities which should be kept under glass (to the elation of many a pupil . . . and perhaps, a teacher). Trigonometry would have to declare a holiday. But not for long, though. . . .

This holiday should not be a very extended respite from numerical work in trigonometry. All trigonometric tables may suffer complete destruction, and this may be a thorough destruction. . . . But, such tables can be reconstructed and recomputed so rapidly that a pupil in the ninth grade can obtain a table of natural trigonometric functions in less time than it takes to teach the solution of quadratic equations. As a matter of fact, this department would be amiss if it did not recommend that the construction of tables of natural trigonometric functions be included in the scope of the teaching of numerical trigonometry. The reasons for this recommendation are deferred until we complete the development of the subject.

The computation of the table of natural trigonometric functions may be confined to the computation of the numerical values of the sine function. Certain numerical values are usually well known to the pupils. These are the numerical values of the sines of 0°, 30°, 45°, and 60°. Thus, we have the following values:

$$\sin 0° = 0.00000, \qquad \sin 30° = 0.50000,$$
$$\sin 45° = \sqrt{2}/2 = 0.70711, \quad \text{and}$$
$$\sin 60° = \sqrt{3}/2 = 0.86603.$$

The reason for carrying these (and all other values) to five significant digits is due to the fact that in further computations some of the significant digits may be

lost. (The rules of approximate computation must be followed.)

The numerical values of the sine functions for small angles (preferably not beyond 4.5°) may be computed in terms of the *radian measures*. Although the notion of a *radian measure* is gradually delegated to college work, it can be easily presented to the pupils as the length of an arc, even if the total numerical value of a radian (that is, 57.29577°) may be avoided. Thus, for small angles (not exceeding 4.5°) the length of an arc does not materially differ from the numerical value of the sine function of the angle which corresponds to that arc. In other words, we may use the formula

$$\sin\theta=\frac{\pi\cdot\theta}{180},$$

where the values of the angle θ lie between the limits of 0° and 4.5° and the radius of the circle associated with the computations of the numerical values of the sine functions is assumed to be 1.

We then obtain the following numerical values of the sine functions

$\sin 0°=0.00000$ $\sin 2°=0.034906$

$\sin 1°=0.017543$ $\sin 3°=0.052359$

$\sin 4°=0.069812.$

It may be pointed out that in the case of the sin 4° the discrepancy between the computed numerical value of sin 4° and actual numerical value of sin 4° is 0. 069812 $-0.069756=0.000056$ which does not materially affect the further computations.

It would be advantageous to compute the numerical values of the sine of 2.5°, 3.5°, and 4.5° also.

The numerical values of the cosine functions for the above angles may be computed by means of the relation $\sin^2\theta+\cos^2\theta=1$.

It may be noted that the numerical values of the cosine functions for the above angles, when computed, do not differ from the actual numerical values of these functions. Thus, we have

$\cos 0°=1.00000$ $\cos 2°0.99939$

$\cos 1°=0.99985$ $\cos 3°=0.99863$

$\cos 4°=0.99756.$

All further computations are performed by means of the relations.

$$\sin 2\theta=2\sin\theta\cos\theta \quad \text{and}$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

When these computations are performed we can be certain of four significant digits in the numerical values of the sine functions, and if some computations require more than one multiplication, the discrepancy in the fourth place of significance (especially, when the computations will go beyond 50° or 55°) will not materially affect the fourth place. We can be definitely certain of three significant digits in the computed numerical value of the sine functions.

From the pedagogical point of view, there is sufficient motivation in a project of the construction of a table of the numerical values of trigonometric functions. The numerical work may be distributed among all the pupils of a single class. This may be a project of the nature of a Junior Institute of Numerical Analysis. Furthermore, this project presents more than ample opportunity for work in arithmetic. From the point of view of evaluation of formulas this project offers something which may be considered more real than the computation of the stress of a beam of steel by means of a formula which is given in an algebra textbook. Furthermore, the pupils are challenged to construct something which they will use in their work in numerical trigonometry.

This department will publish any comments of teachers who are brave enough to accept this challenge to try out the above recreational procedure. This is the teachers' department. Their experiences in connection with the above recreational activity will certainly be gratefully shared by other teachers.

REFERENCES FOR MATHEMATICS TEACHERS

Edited by WILLIAM L. SCHAAF
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Testing and Evaluation in Mathematics

During the past few years, considerable progress has been made in improving the quality of tests and evaluation procedures, particularly along the following lines:

(1) a notable shift in emphasis from measuring habits and skills to a recognition and appraisal of understanding and power;
(2) a more determined effort to evaluate the quality of problem-solving reasoning, the use of judgment, critical thinking, attitudes, and appreciations;
(3) a frank admission that "paper and pencil" techniques must be supplemented by observation, discussion and interviews in order to secure valid evaluation.

Teachers should earnestly seek to improve the testing program in these directions, and may perhaps find some helpful suggestions in the bibliographic data below.

1. GENERAL CONSIDERATIONS

Ballard, R. M. "Construction of Objective Tests in Mathematics." *National Mathematics Magazine*, 1941, 16: 150–56.

Cattell, R. B. "Personality Traits Associated with Abilities, II: With Verbal and Mathematical Abilities." *Journal of Educational Psychology*, 1945, 36: 475–86.

Chrisman, E. B. "Let's Examine Our Tests." *School Science and Mathematics*, 1947, 47: 684–87.

Cox, H. M. "Pre-study Examinations in Mathematics." *National Mathematics Magazine*, 1943, 17: 351–59.

Cox, H. M. "What Are the Administrative and Guidance Uses of Mathematics Examinations?" *American Mathematical Monthly*, 1941, 48: 317–19.

Erskine, Wm. H. "The Use of Index Numbers in Evaluation." *National Mathematics Magazine*, 1942, 16: 252–58.

Grossman, G. W. "Adjusting Low Scores." MATHEMATICS TEACHER, 1945, 38: 81–82.

Hartung, M. L. "A Forward Look at Evaluation." MATHEMATICS TEACHER, 1949, 42: 29–33.

Hartung, M. L. "Some Problems of Evaluation." MATHEMATICS TEACHER, 1938, 31: 175–82.

Jonah, H. F. S., and Keller, M. W. "Some Variations of the Multiple-choice Question." *American Mathematical Monthly*, 1945, 52: 1–6.

Kenney, J. J. "Mathematics in Group Intelligence Tests." *Journal of Educational Research*, Oct. 1950, 44: 129–33.

Krathwohl, W. C. "Constructing a Mathematics Achievement Test." *Journal of Engineering Education*, Nov. 1949, 40: 178–86. Bibliography.

Plumlee, L. B. "Verbal Components in Mathematical Items." *Educational and Psychological Measurement*, 1949, 9 no. 4: 679–84.

Reklis, V. M. "Use of Medians in a Testing Program in Mathematics." *National Mathematics Magazine*, 1942, 16: 345–49.

Syer, Henry. "A Cooperative Plan to Share Objective-Type Test Items." MATHEMATICS TEACHER, 1951, 44: 302–6.

Teller, J. D. "Some Newer Forms of the Recognition Test." *School Science and Mathematics*, 1944, 44: 859–63.

2. TESTING IN SECONDARY MATHEMATICS

Bromley, A. and Carter, G. C. "Predictability of Success in Mathematics." *Journal of Educational Research*, Oct. 1950, 44: 148–50.

Butler, C. H. and Wren, F. L. *The Teaching of Secondary Mathematics*. Rev. edition. N. Y., McGraw-Hill, 1951, pp. 201–26.

Challman, Mildred. "The Retention of Arithmetic and Algebra in Relation to Achievement in Geometry." MATHEMATICS TEACHER, 1946, 39: 77–79.

Christofferson, H. C. "Appraising Mathematical Skills of Our Seniors." *Ohio Schools*, 1946, 24: 104–5.

Davis, D. R. *The Teaching of Mathematics*. Cambridge, Mass. Addison-Wesley Press, 1951. Gives a partial list of standardized tests in arithmetic, algebra, geometry and trigonometry.

Davis, R. A. and Henrick M. "Predicting Accomplishment in Plane Geometry." *School Science and Mathematics*, 1945, 45: 403–5.

Douglass, H. R. "The Prediction of Pupil Success in High School Mathematics." MATHEMATICS TEACHER, 1935, 28: 489–504.

Gibney, E. F. "Aptitude Tests in Relation to the

Teaching of Plane Geometry." MATHEMATICS TEACHER, 1949, 42: 181–86.

Glaubiger, I. "Causes and Remedies for Pupil Failure in High School Mathematics." *High Points*, Sept., 1941, 23: 26–31.

Hartung, M. L. and Fawcett, H. P. "The Measurement of Understanding in Secondary-School Mathematics." *National Society for the Study of Education*, 45th Yearbook, Part I, 1946, pp. 157–74.
Discusses evaluation of outcomes under the following headings: number and computation, geometric form and space perception, graphic representation, elementary analysis, logical thinking, relational thinking, symbolic representation.

Hastings, J. T. "Testing Junior High School Mathematics Concepts." *School Review*, 1941, 49: 766–76.

Hendricks, B. C. "Why Not Abolish Tests?" *School Science and Mathematics*, 1947, 47: 114–16; "For What Shall We Test?", 47: 203–6; "How Shall We Test?", 47: 322–25; How Good Are Our Tests?", 47: 470–74; "How to Improve Our Tests," 47:554–59.

Hickox, J. G. and Van Metre, R. "Further Analysis Made of Mathematics Screening Test for Seniors." *Ohio Schools*, 1946, 24: 370–71.

Holzinger, K. J. and Swineford, F. "The Relation of Two Bi-factors to Achievement in Geometry and Other Subjects." *Journal of Educational Psychology*, 1946, 37: 257–65.

Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics. "The Place of Mathematics in Secondary Education," 15th Yearbook of the *National Council of Teachers of Mathematics*. New York, Teachers College, Columbia University, 1940, pp. 162–86.

Kidd, H. C. "Overcoming Difficulties in Mathematics." *School* (Elementary Edition), Jan. 1944, 32: 427–29.

Kraft, Ona. "Methods Used in the Selection of Pupils for the Study of Algebra and Geometry in Cleveland." MATHEMATICS TEACHER, 1946, 39: 236–39.

Layton, R. B. "A Study of Prognosis in High School Algebra." *Journal of Educational Research*, 1941, 34: 601–5.

Lee, H. L. "Statistical Analysis of an Elementary Algebra Test." (Abstract.) *Ohio University, Abstracts of Masters' Theses*, 1940: 30.

Marie, Sister Noel. "Rating Papers in Mathematics." *Catholic School Journal*, 49: 161.

Massing, O, "Testing to Insure Mastery of Fundamentals in Mathematics." *A.T.A. Magazine*, Jan. 1945, 25: 37–38.

Meder, E. M. and Eagle, E. "Measurement and Evaluation in the Teaching of Mathematics and Science." *Review of Educational Research*, Oct. 1948, 18: 350–63.

Morris, L. "Prognosis in First-year Algebra." *Ohio Schools*, Jan. 1940, 18: 20–21.

Northrup, G. J., Pingry, R. E., and Winsor, A. L. "A Functional Item Analysis of Group Performance in General Mathematics." *Journal of Experimental Education*, 1950, 18: 279–85.

Ohlsen, M. M. "Control of Fundamental Mathematical Skills and Concepts by High School Students." MATHEMATICS TEACHER, 1946, 39: 365–71.

Orleans, J. B. "Evaluation in Secondary Mathematics." *Secondary Education*, 1942: 10, 314–17.

Orleans, J. B. "A Study of Prognosis of Probable Success in Algebra and in Geometry." MATHEMATICS TEACHER, 1934, 28: 165–80; 225–46.

Poppen, Henry. *A Factor-analysis Study of Prognostic Tests in Algebra*. Nashville, Tenn., Contribution to Education No. 240, George Peabody College for Teachers, 1950.

Potter, M. A. "Testing." MATHEMATICS TEACHER, 1951, 44: 83–86. Includes a brief list of published tests in arithmetic and high school mathematics.

Progressive Education Association. *Mathematics in General Education*. Report of the Mathematics Committee of the Commission on Secondary School Curriculum. New York, Appleton-Century-Crofts, 1940, pp. 338–82.

"Report on Kansas Mathematics Test Number Two in Arithmetic and Algebra." *Bulletin, Kansas Association of Teachers of Mathematics*, October 1941, 16: 3–9.

"Report on Kansas Mathematics Test No. 3 in Arithmetic and Algebra." *Bulletin, Kansas Association of Teachers of Mathematics*, October 1942, 17: 3–6.

Rosilda, Sister M. "Is an I.Q. an Index to Algebra Ability?" *Journal of Educational Research*, 1951, 44: 391–93.

Russel, O. R. "Geometry Examination for Grade 12." *School* (Secondary Edition), April-May 1942, 30: 685–90; 784–86.

Sears, V. P. *A Study of the Evaluation of Learning in High School Algebra*. Doctor's Thesis. New York, Teachers College, Columbia University, 1949. 152 p. (Typewritten.)

Sprague, J. "Diagnostic Testing to Improve Mathematical Ability in Grade X." *School Review*, 1939, 47: 431–38.

Sueltz, B., Boynton, H. and Sauble, I. "The Measurement of Understanding in Elementary School Mathematics." *National Society for the Study of Education*, 45th Yearbook, Part I, 1946, pp. 138–56. Concerned chiefly with arithmetic; directs attention to two principle methods of evaluation, viz., by use of paper and pencil exercises, and by observation, discussion and interview.

Thompson, R. "Diagnosis and Remedial Instruction in Mathematics." *School Science and Mathematics*, 1941, 41: 125–28.

Thorndike, Albert. "Correlation between Physics and Mathematics Grades." *School Science and Mathematics*, 1946, 46: 650–53.

Traxler, Arthur. "A Testing Program for Mathematics at the Secondary-School Level."

MATHEMATICS TEACHER, 1946, 39: 303–13. Bibliography.

Truesdale, Beatrice. "Prediction of Success or Failure in Geometry." *California Mathematics Council Bulletin*, Nov. 1947, 5: 14–16.

Vasudevan, S. "Diagnostic Tests for Optional Mathematics." *Teaching* (Bombay), June 1940, 12: 173–75.

Volpel, M. C. "Results of an Inventory Test." *School Science and Mathematics*, 1942, 42: 188–91.

3. TESTING IN COLLEGE MATHEMATICS: PLACEMENT TESTS

"An Addition to the Mathematics Attainment Test of the College Entrance Examination Board." MATHEMATICS TEACHER, 1942, 35: 3–4.

Boeker, M. D. *Status of the Beginning Calculus Students in Precalculus College Mathematics.* Teachers College, Columbia University, 1947. 83 p.

Boyd, E. N. "Diagnostic Study of Students' Difficulties in First Year College Work." (Abstract). *Teachers College Record*, 1941, 42: 344–45.

Chittenden, E. "Report of the Committee on Tests." *American Mathematical Monthly*, 1940, 47: 290–301.

Dichter, M. R. "Relationship between Aptitude Scores and College Grades in Mathematics and Science." *American Association of Collegiate Registrars*, 1937, 13: 16–37.

Douglass, H. R. and Michaelson, J. H. "The Relation of High School Mathematics to College Marks and of Other Factors to College Marks in Mathematics." *School Review*, 1936, 44: 615–19.

Frederiksen, N. "Predicting Mathematics Grades of Veteran and Non-veteran Students." *Educational and Psychological Measurements*, 1949, 9: 173–88.

Habel, E. A. "Experiment in the Diagnosis and Remedy of Errors of College Freshmen in Arithmetic and Radicals." *School Science and Mathematics*, 1951, 51: 105–13.

Harper, Floyd. "An Experiment in Selecting Students According to Ability and Measuring their Achievement by Common Examinations." *National Mathematics Magazine*, 1944, 19: 27–32.

Held, O. C. "College Mathematics Placement Test." *Journal of High Education*, Jan. 1942, 13: 39–40.

Keller, M. W. and Shreve, D. R. "Abilities of University Students in Freshman Mathematics." *School Science and Mathematics*, 1942, 42: 38–46.

Keller, M. W., et al. "Diagnostic Testing Program in Purdue University." *American Mathematical Monthly*, 1943, 50: 85–90.

Keller, M. W., et al. "Mathematics Diagnostic Testing Program in Purdue University; the Error Analysis of the Number Technique Test." MATHEMATICS TEACHER, 1942, 35: 8–14.

Keller, M. W.; et al. "Report on a Diagnostic Testing Program at Purdue University; the Number Technique Ability of Students in Beginning Courses in Mathematics." MATHEMATICS TEACHER, 1940, 33: 321–24.

Keller, M. W. and Jonah, H. F. S. "Measures for Predicting Success in a First Course in College Mathematics." MATHEMATICS TEACHER, 1948, 41: 350–55. Bibliography.

Kilbridge, J. T. "How Different Are the Several Objectives of a Course of Study in Mathematics?") *College and University*, Jan. 1948, 23: 201–11.

Kossack, C. F. "Mathematics Placement at the University of Oregon." *American Mathematical Monthly*, 1942, 49:234–37.

Robertson, J. F. "Some Phases of the Mathematics Testing Program at the Iowa State College." MATHEMATICS TEACHER, 1943, 36: 296–302.

Rutt, Norman. "Grades and Distribution." *National Mathematics Magazine*, 1943, 18: 120–26.

Scott, W. M. and Gill, J. P. "A Prediction of Pupil Success in College Algebra." MATHEMATICS TEACHER, 1941, 34: 357–59.

Stright, I. L. "Measurement of Success in College Mathematics." MATHEMATICS TEACHER, 1945, 38: 302–5.

Sueltz, Ben. "Mathematical Understandings and Judgments Retained by College Freshmen." MATHEMATICS TEACHER, 1951, 44: 13–19. Stresses the limitations of "paper and pencil" technique of testing; gives illustrations of test items seeking to reveal mathematical understanding and judgment.

Trimble, H. C. and Babcock, L. E. "Experience in Evaluation." *Junior College Journal*, March 1950, 20: 403–11.

Wolfe, Jack. "Mathematical Skills of College Freshmen in Topics Prerequisite to Trigonometry." MATHEMATICS TEACHER, 1941, 34: 164–70.

4. TESTING MATHEMATICAL THINKING

Ebert, R. S. "Generalization Abilities in Mathematics." (Abstract of thesis). *Journal of Educational Research*, 1946, 39: 671–81.

Fawcett, H. P. "Foust-Schorling Test of Functional Thinking in Mathematics." *Educational Research Bulletin*, May 1945, 24: 140.

Hartung, M. L. "Evaluating Appreciation of the Cultural Values of Mathematics." *School Science and Mathematics*, 1937, 37: 168–81.

Rosskopf, M. F. "Present State of Evaluation of Critical Thinking in Algebra and Geometry." MATHEMATICS TEACHER, 1950, 43: 143–48.

Ullsvik, B. R. "An Attempt to Measure Critical Judgment." *School Science and Mathematics*, 1949, 49: 445–52.

Ulmer, Gilbert. "Tests for Critical Thinking." *Bulletin Kansas Association of Teachers of Mathematics*, 1945, 19: 40.

BOOK SECTION

Edited by JOSEPH STIPANOWICH
Western Illinois State College, Macomb, Illinois

BOOKS RECEIVED

Elementary School

Growth in Arithmetic, Grade 3, by John R. Clark, Charlotte W. Junge, and Harold E. Moser. Cloth, v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

Growth in Arithmetic, Grade 4, by John R. Clark, Charlotte W. Junge, and Harold E. Moser. Cloth, v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

Growth in Arithmetic, Grade 5, by John R. Clark, Harold E. Moser, and Charlotte W. Junge. Cloth, v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

Growth in Arithmetic, Grade 6, by John R. Clark, Harold E. Moser, and Charlotte W. Junge. Cloth, v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

Growth in Arithmetic, Grade 7, by John R. Clark, Rolland R. Smith, and Harold E. Moser. Cloth, v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

Growth in Arithmetic, Grade 8, by John R. Clark, Rolland R. Smith, and Harold E. Moser. v+314 pages, 1952. World Book Company, Yonkers-on-Hudson, New York. $2.12.

College

Advanced Mathematics

Finite Deformation of an Elastic Solid, by Francis D. Murnaghan, Instituto Tecnólogico de Aeronáutica, Brazil. Cloth, 140 pages, 1951. John Wiley and Sons, Inc., 440 Fourth Avenue, New York 16, N. Y. $4.00.

Introduction to Mathematical Thinking, by Friedrich Waismann, Oxford University; trans. by Theodore J. Benac, U. S. Naval Academy. Cloth, x+260 pages, 1951. Frederick Ungar Publishing Co., 105 East 24th Street, New York 10, N.Y. $4.50.

Miscellaneous

Dictionary of Mathematical Sciences, Vol. I., German-English, by Leo Herland. Cloth, 235 pages, 1951. Frederick Ungar Publishing Co., 105 East 24th Street, New York 10, N.Y. $3.25.

Residence and Migration of College Students, 1949-50, by Robert C. Story, Federal Security Agency, Washington, D. C. Paper, vi+61 pages, 1951. Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. $0.25.

REVIEWS

Plane Trigonometry, H. M. Dadourian. Cambridge, Mass., Addison-Wesley Press, 1950. x+109 pp.; tables 84 pp., $3.50.

This is an interesting and well-constructed text; the formation is good, the figures excellent. Although the book is brief, it contains all the essential topics, an excellent chapter on applications to physics, and a chapter on logarithms. Numerical problems are solved as literal problems first, and the expressions for the required magnitudes are checked for validity and interesting special cases. The reduction formulas are presented in a concise manner requiring a minimum of memorizing for the student. There is an abundance of good problems; answers to the odd-numbered ones are given. The critical instructor will occasionally supplement considerably the text discussions. Not to be found in this text are the usual comments about division by zero and the fact that the listed definition of, say, tan A, does not define tan 90°. It is stated that an identity is a valid statement for all values of the literal symbols involved. A more critical analysis seems desirable, particularly for the student who may soon be studying indeterminate forms. In the chapter on logarithms a general rule is given regarding computation with approximate data; this rule is violated repeatedly throughout the text.—LAWRENCE A. RINGENBERG, Eastern Illinois State College, Charleston, Illinois.

Practical Mathematics, Part IV, Trigonometry and Logarithms (Fifth ed.), C. I. Palmer and S. F. Bibb. New York, McGraw-Hill Book Company, 1951. xii+193 pp., $2.60.

As a whole, the book is well done. It has an extremely good group of practical problems and adequate examples. The book would not be easily used by the average high school student but for the somewhat more mature mind and especially for those with some practical experience and acquaintanceship with the terminology, it is an excellent text. The book should provide the high school advanced algebra and trigonometry teacher with an excellent source of enrichment materials. The book is a bit mechanical in a few respects, i.e., two separate rules for determining the characteristic of a logarithm; but, in general, it lays far more em-

phasis on understandings and the "why" aspect than most books of its nature.

There is a four-place logarithm table and the trigonometric functions, natural and logarithmic for four functions, are included to the nearest ten minutes. Answers to all problems are contained at the end of the book.—ROBERT V. BELDING, Thomas Carr Howe High School, Indianapolis, Indiana.

Analytic Geometry (Second ed.), John W. Cell. New York, John Wiley and Sons, 1951. xii+326 pp., $3.75.

This is a comprehensive textbook in analytic geometry, generously supplied with problems and intended for students who will continue with mathematics, pure or applied. The book contains sufficient material for a five-hour, one-semester course.

The two basic objectives of analytic geometry are clearly stated: to learn to draw graphs of given equations and to learn to find equations of certain curves. The author has placed his emphasis on these two aims and away from memory work. He has explained each topic clearly and there are problems, illustrations and summaries to aid the student in his mastery of the material. In many cases more than one approach to a problem is made, thus giving the student an opportunity to know and use different methods for different solutions.

The chapter headings are: 1. Elementary Concepts of Analytic Geometry, 2. The Locus Derivation and the Straight Line, 3. Curve Sketching, 4. Conics, 5. Transcendental Curves, 6. Polar Coordinates, 7. Parametric Equations, 8. Empirical Equations, 9. Planes and Lines in Solid Analytic Geometry, 10. Surfaces in Solid Analytic Geometry. A review of basic material follows which gives a ready reference to geometry, algebra and trigonometry.

We have in this book a good text from which to teach analytic geometry as a course separate from the calculus. Students will find it clear and very readable and the instructor can enjoy a wide variety of problems and methods.—CAROLINE A. LESTER, New York State College for Teachers, Albany, New York.

Analytic Geometry, Raymond D. Douglass and Samuel D. Zeldin. New York, McGraw-Hill Book Company, 1950. ix+216 pp. $2.75.

The text is simple and direct in its approach to the subject and contains a minimum of properties of the various aspects treated. The purpose of the text is to give only that analytic geometry that is essential to the study of differential and integral calculus, and to cover it in the least possible time. Much that is found in the usual analytic geometry text is omitted, for example dividing a line segment is restricted to positive ratios, the normal form for a straight line is omitted; in the conics, treatment of chords, diameters, focal radii and latus rectum, are missing. Each conic is treated separately and concisely; there is no general treatment except for the general second degree equation in x and y. Here translation of axis is given as a process, but no general formula is developed for eliminating the xy term. The use of the discriminant C^2-4AB for determining the nature of the curve is stated without proof. There is a brief treatment of lines and planes in space and the common surfaces. Stress is placed on polar coordinates, parametric representation and the plotting of curves.

The study of text presupposes the study of trigonometry and algebra to include determinants of the third order. It has a wealth of pertinent exercises for the student. The illustrations are good and presented clearly. It is a good text for a student. There are a few statements open to criticism. To write

$$\frac{m_2-m_1}{1+m_2m_1}=\infty$$

when $1+m_2m_1=0$, without the use of limits concepts can be puzzling, especially to students who have been told division by 0 is not permitted. On page 47 there is evidently a slip in using 1 instead of 5 for the radius of the circle. On page 25 Ex. 11 in

$$\frac{x^2}{m^2}+\frac{y^2}{m^2-4}=1$$

the student is unprepared to answer when $m=0$, or ± 2. This may be due to the author's desire to avoid complicating issues and to get a working basis in analytic geometry. In this they have succeeded.—HOWARD F. FEHR, Teachers College, Columbia University, New York.

Brief Course in Analytics, Revised Edition, M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Co., 1951. xi+224 pp., $2.40.

This book represents a revision of the 1940 edition in which the content and arrangement of the topics of the earlier edition have been maintained by the authors. However the number of exercises included has been increased and the numerical data of the problems has been changed. The material presented is that of the traditional course but the explanations are complete and written in language which should be easily understood by the students. Many figures are included and in most cases are drawn on coordinate paper which should help to establish the relationships involved in the locus being studied. The material on solid analytic geometry is brief but well chosen, covering the basic ideas necessary for a study of calculus.

The authors state that in this edition, as in the previous one, their hope has been "to help the student to develop his powers of reasoning." Problems included are intended to lead from the simple to the difficult thus building up the student's confidence and his problem-solving ability.

The textbook is designed for classes which meet three hours per week and is particularly adaptable for courses where teacher supervision is at a minimum.—HERBERT HANNON, Western Michigan College of Education, Kalamazoo, Michigan.

The Teaching of Secondary Mathematics (Second ed.), Charles H. Butler and F. Lynwood Wren. New York, McGraw-Hill Book Co., 1951. xiv+550 pp., $4.75.

Anyone desiring a book that is broad in its coverage of the field of secondary mathematics will find it worth their time to investigate this presentation. It is packed with potential value for college people who are the instructors of future teachers of mathematics or those having returning teachers in their classes, as the book presents a sound interpretation of techniques available for instructing on a secondary school level.

Teachers of high school mathematics will find this to be a good supplement to their reading materials for it contains a condensed version of many studies that have been conducted concerning various approaches to mathematics instruction. The authors have done a commendable job of discussing various techniques that have been, are and should be used in high school mathematics teaching.

Specifically, the material considers the place and function of mathematics in secondary education from a historical, contemporary and rather prophetic vein. Much consideration is given to the improvement of instruction in secondary mathematics and some of the topics considered are: instructional materials, aids to teaching, motivation, methods and evaluation of instruction, and the professionalism of teachers of mathematics. The latter material is concerned with the teaching of special subject matter of the secondary school which includes arithmetic, algebra, geometry, trigonometry and calculus.—Roderick C. McLennan, Arlington Heights Township High School, Arlington Heights, Illinois.

The Teaching of Mathematics, David R. Davis. Cambridge, Mass., Addison-Wesley Press, 1951. xv+415 pp., $4.50.

Introduced by the author as a textbook for a one-semester course in the techniques of teaching mathematics, the book covers both educational method and course content for grades 1 to 14. By succinctness of treatment the author has succeeded in packing a remarkable amount of information, useful to the mathematics teacher, into a volume of average size.

There is an approximately even division between the amount of space devoted to purely didactic considerations and the pages on mathematical theory used in primary, secondary and junior college mathematics. The author has endeavored to include all possible aspects of the mathematics teacher's professional life and gives a comprehensive discussion to the contents of traditional as well as non-traditional curricula in mathematics. Each chapter is followed by a list of suggested projects and questions for the student's benefit. A long list of classified references, covering both books and articles in periodical literature, is appended.

On certain debatable questions of teaching technique the author has taken a thoroughly up-to-date position. Many of the recommendations of the National Council are supported by implication. The over-all impression of the book is that of a happy balance between sound conservatism and constructive reform. The reviewer believes the book to be a good text for teacher training courses and a very useful reference volume for the mathematics teacher's professional library.—Paul R. Neureiter, State University Teachers College, Geneseo, New York.

Space and Geometry, Aaron Bakst. New York, University Book Store, 1950. 80 pp. $.90.

This monograph presents material which, in the author's opinion, is a *must* for the successful teaching of geometry. The mathematics teacher, he says, needs "some definite background material which will throw some clear and unobstructed light on the nature of the subject." Whether geometry is approached as a "purely abstract system" or as a study of the "nature and properties of the elements which are assigned to this field . . . certain space-properties need to be recognized." It would be fallacious, the author states, to deny that Space has certain physical properties when certain phases of geometry are involved, but it is also fallacious to think of Space in physical terms exclusively when geometry is concerned.

Dr. Bakst develops and examines the notion of Space along purely mathematical lines; he concerns himself with those properties of Space which are in conformity with the fundamental principles which are accepted for a given system of geometry. He examines various sets of postulates, Euclidean, Lobatschevskian, and Riemannian geometries and their relative ideas of Space; he examines geometric transformations. He emerges from this "inquiry concerning the relations of postulational procedures and postulated elements to the domain within which they are supposed to be operative" with a method which "enables us to formulate a unified approach to Space as well as a unified approach to geometry."

Geometry teachers in training and those in service will find *Space and Geometry* an excellent "extender" and "unifier" of geometric ideas.—Elinor B. Flagg, Illinois State Normal University, Normal, Illinois.

The Theory of Valuations, O. F. G. Schilling. New York, American Mathematical Society, 1950. viii+253 pp. $6.00.

This volume is number four in the Mathematical Surveys series of the American Mathematical Society. The purpose of this series is to publish monographs each of which is a systematic presentation of a topic of major interest in present-day mathematics. As such, these monographs are intended for those who have achieved a rather advanced stage of mathematical maturity. This volume is no exception. In fact, since a rather extensive subject is covered in a fairly small number of pages, the presentation

(*Continued on page 231*)

WHAT IS GOING ON IN YOUR SCHOOL?

Edited by JOHN R. MAYOR *and* JOHN A. BROWN
The University of Wisconsin, Madison, Wisconsin

MAPPING THE SCHOOL DISTRICT

(A Trigonometry Problem)

Ensley High School chose as its theme for the semester "Appreciation of Our Environment." Every teacher in each department carried out the theme in her work.

We, the Trigonometry Class, mapped the Ensley High School district as our project. Two large maps of the district were made. The first, a geographical map, shows the location of Ensley High School, the elementary schools, parks, the steel mill, Ensley's important buildings, and the routes of transportation. It also shows Ensley mapped in comparison to the city of Birmingham. The second map shows how the district was divided into triangles to enable us to find the area trigonometrically.

The idea was born; the work began.

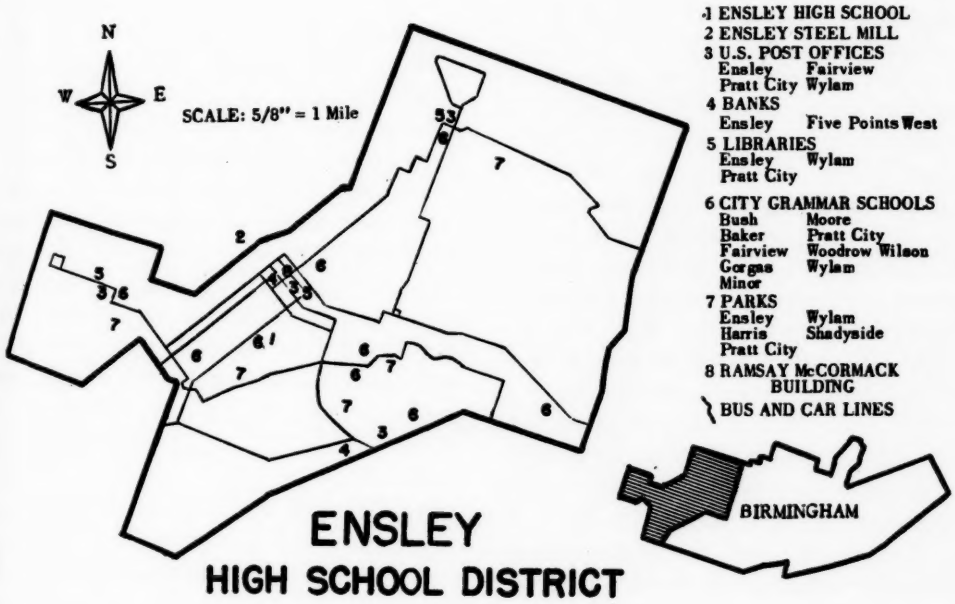
Six of us in the class were planning to be engineers. We were made the general chairmen of the project; that is, we engineered the job. Individually, in groups, with and without the teacher, we made many trips gathering facts and information.

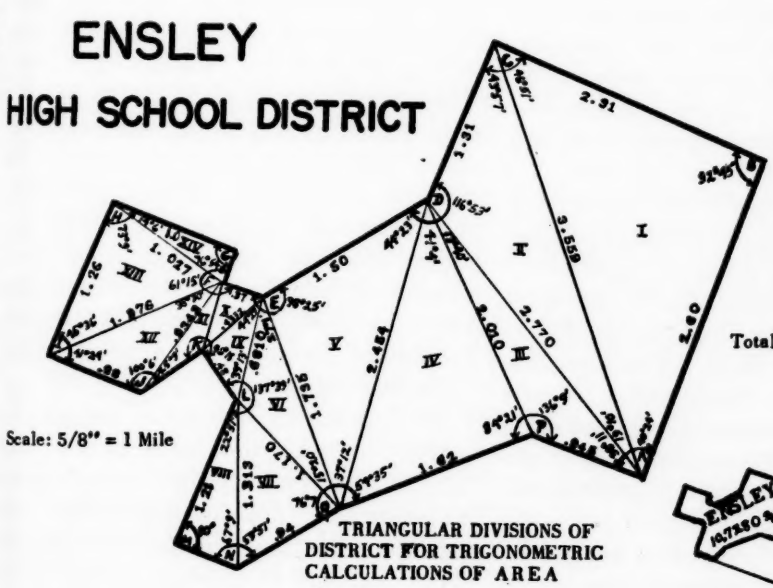
First we went to the Research Department of the Birmingham Board of Education. There we obtained a map of the city of Birmingham. We were told that it would be almost impossible to get the original boundary lines of the Ensley district, because Ensley, like Topsy, just grew. In fact Ensley has grown by annexation since this project was finished. We did find that in 1784 the United States government inaugurated a "System of Surveying Public Lands," but the system has been modified since by Acts of Congress that require public lands to be divided by north and south lines according to a true meridian, which in north Alabama passes through Huntsville, and by crossing them at right angles so as to form townships six miles square. We also found and know that it is mathematically impossible, since meridians converge, for townships to be six miles square and at the same time for the lines to conform to true meridians. We did find that Ensley is located in Range 3 and 4, Townships 17, 18, 19.

Next, we went to the Office of the City Engineer at the City Hall, where we looked through maps and maps, but we could not get our needed corners, lines, or angles. Again we were told that getting these would be an almost impossible and hardly probable task, as the original and successive land grants and deeds were written in the phraseology of the period, some in reference to monuments no longer in existence, or streams whose courses have been altered, or trees that have been felled, or even surveyors' markers that have been removed. So we gave up the idea of getting the exact measurements of the boundary lines with their included angles.

We went to the Jefferson County Engineering Department, Jefferson County Court House. There we met a young engineer, who, catching the enthusiasm for the project, volunteered to get the information for us by scaling the map and referring to other maps and records on file at the Court House.

So, at long last, we had a map of the Ensley High School District with the boundary, dimensions, and the angles placed upon it. All except two of the original boundary lines were straight lines.





AREAS OF TRIANGLES

I	2.9995 Sq. Miles
II	1.618
III	.658
IV	1.62
V	1.2873
VI	.2684
VII	.5337
VIII	.3086
IX	.1033
X	.0753
XI	.1159
XII	.3958
XIII	.6192
XIV	.1250

Total Area = 10.7280 Sq. Miles

On our map we straightened Arkadelphia Road, as well as the boundary line between Wylam and Pratt City. The areas which were added or cut off approximately counterbalanced each other. This enabled us to cut the district into fourteen triangles which we numbered I to XIV and lettered from A to P. Each triangle was chosen by one student, who was responsible for solving, checking, drawing to scale and then checking again. In solving the triangles we used the laws of sines, cosines, tangents and the Pythagorean Theorem.

When triangle VI was formed, none of its sides or angles were scaled on the map. We had to begin at both ends of the map and work towards that triangle. After solving the surrounding triangles, we had the three sides of Triangle VI. Using the three sides we found the angles and the area. Due to the human element and slight inaccuracies of measuring instruments, logarithm tables, and curvature of the earth, the hills and valleys over the area, there will be variations in the calculations. However, with our data thus obtained, these problems were solved, checked by the students and by the teacher and finally they were used as classroom and test problems.

We interviewed and obtained material from the Research Department of the Birmingham Board of Education; the City Engineer's Office, City Hall; The Jefferson County Engineering Department, Jefferson County Court House; The Land Department and the Engineering Department of the Tennessee Coal and Iron Company; the Transportation Department of the Birmingham Electric Company.

We learned much. We met many fine, busy, successful businessmen who graciously gave their time and talent to encourage the young mathematicians. Their inspiration was contagious.

Rosa W. Blackwell
Ensley High School
Birmingham, Alabama

Mental Arithmetic in the Junior High Schools of Cleveland

Plans for Teaching Mental Arithmetic

Believing that the schools should cultivate the pupils' ability to solve simple problems without the use of a pencil and paper, a definite plan for such instruction has been put into practice in the junior high schools of the city. Pupils in the seventh grade study two different types of lessons in mental arithmetic as a part of the regular course in mathematics. Each semester, eight lessons consisting of problems presented in the form of short dramatic episodes are broadcast city-wide from radio Station WBOE. A description of these lessons will be given later in this paper.

In addition to the lessons given over the radio, a mimeographed pamphlet containing problems to be solved mentally, prepared by a committee of teachers, is being used in all seventh grade classes. Similar lessons consisting of problems that are somewhat more difficult are used in eighth grade classes. Each of these lessons consists of twenty problems. They are read to the pupils by the teacher. The pupils must give careful attention, select the facts needed to solve the problem and perform the necessary computations mentally, using a pencil and paper only for writing the answer. A reasonable amount of time, depending upon the ability of the pupils in the class, is allowed for solving the problem and recording the answer.

The problems in these lessons are similar to those that arise in the daily life of the junior high school pupil. Furthermore, they involve simple computations that can be performed mentally. They deal with such things as figuring the cost of a luncheon at the cafeteria, finding the difference in cost of articles purchased, figuring the amount of change that should be received from a purchase, and the like. Several problems typical of those included in these lessons follow:

1. A grocery store sells a certain brand of canned corn at 19 cents a can or three cans for 52 cents. How much is saved by buying three cans at a time rather than one at a time?
2. Mary served at a luncheon for 2 hours and 30 minutes. She was paid 60 cents an hour. How much pay did she receive?
3. At what time must you start from home to catch a train that leaves at 7:15 p. m. if the trip takes 25 minutes?
4. Robert drove his father's car to a service station and bought gasoline for $2.30 and oil for 35 cents. How much change should he have received from a five-dollar bill?
5. Ray is figuring the cost of attending a football game. These are the expenses he has listed: ticket, 50¢; bus fare, 36¢; sandwich, 15¢. What are the total expenses?

The Use of The Radio in Teaching Mental Arithmetic

During the past year a series of lessons in mental arithmetic for seventh grade pupils has been presented over the radio. The problems included in the lessons had been made to resemble as nearly as possible those that actually arise in everyday experiences. They are presented orally in dramatic episodes by a group of boys and girls from the junior high workshop at Station WBOE. These dramatizations create imaginatively in the minds of the listeners life situations involving numbers that take place in school, in the home, at the store or at work. The pupils in the classrooms listen to the dramatizations and then work the problems mentally, using pencil and paper only for recording the answers.

The dramatic episodes that follow are typical of those included in the lessons. The problems involved are representative of those that arise in the affairs of everyday life. The figuring that must be done to solve them is the kind usually done without the use of a pencil and paper. Each lesson consists of twelve problems, the first being preceded by a short introduction by the program announcer.

ANNOUNCER: Now here is the first problem. Mary is telling James about her uncle, who has just arrived from the East. Listen closely, think quickly, and . . . Get the Answer—Right!

JAMES: (*on cue*) Did your uncle get here from Boston yesterday, Mary?

MARY: From Springfield; not Boston. We met him at the airport yesterday afternoon.

JAMES: Oh, he flew! How long did it take him on the plane?

MARY: Well, it's six hundred miles to Springfield.

JAMES: I know, but how long did the trip take?

MARY: Well, the plane came in at the airport at five-forty-five yesterday.

JAMES: And when did it leave Springfield?

MARY: At two-fifteen.

JAMES: Then it took him . . .

ANNOUNCER: Problem one: how long did it take to fly from Springfield to Cleveland?

SOUND: (*after five seconds . . . on cue*) GONG

ANNOUNCER: Time's up, and we'll go right on to problem two. Here we have Helen and Harry. They're discussing the ticket sale for a football game. Listen closely, think quickly, and . . . Get the Answer—Right!

HELEN: (*on cue*) Oh, Harry, I've been looking for you. How's the ticket sale coming along for Friday's game?

HARRY: Pretty good, I think. I was just talking to two of the salesmen.

HELEN: Well how many have we sold?

HARRY: I don't know exactly; I'll have to figure it up. Lend me a pencil, will you please?

HELEN: I don't have a pencil. We don't need one. We can figure it in our heads. How many did Stanley sell?

HARRY: Twenty-seven.

HELEN: . . . and John?

HARRY: Thirty-six.

HELEN: . . . and Mildred?

HARRY: Ah, uh, let's see. She sold eighteen.

HELEN: And who was the other salesman?

HARRY: George. He sold thirteen.

HELEN: So altogether that makes . . .

HARRY: That's a total of . . .

ANNOUNCER: Problem two—How many tickets were sold altogether?

SOUND: (*after five seconds . . . on cue*) GONG

ANNOUNCER: Time's up, and here is problem three. Jane has found a bargain in some material for a costume. Listen closely, think quickly, and . . . Get the Answer—Right!

JANE: (*on cue*) Look at these two samples, Bertha. Which material do you like better?

BERTHA: HMMMM. What are you using the material for?

JANE: To make my costume for the play next week.

BERTHA: I think either one will be all right for that. Which is cheaper?

JANE: Now let me see. I need four-and-a-half yards for the costume . . .

BERTHA: Un hunh . . .

JANE: And this piece from a bolt costs fifty cents a yard . . .

BERTHA: That's . . .

JANE: And this other piece is from a remnant of four-and-three-quarter yards . . .

BERTHA: Oh!

JANE: And the remnant costs two dollars.

BERTHA: Why then by buying the remnant you'd save . . .

ANNOUNCER: Problem three—How much will Jane save by buying the remnant?

SOUND: (*after five seconds . . . on cue*) GONG

ANNOUNCER: Time's up.

Increasing Pupil Interest Through Contests

Each semester a series of eight lessons, such as those described previously, are presented over the radio. These lessons, intended for seventh grade pupils, are used by the classroom teachers as a part of the regular work in mathematics. Thus, all pupils in the seventh grade have an opportunity of profiting from this type of instruction. To increase the interest of pupils and to encourage them to do their very best in this subject, a team contest and an individual pupil contest are held each semester. The first seven lessons of the series are used for the team contest and the final program is used for the individual pupil contest.

Each junior high school entering the contest sends one seventh grade pupil to Station WBOE for each of the seven programs. Here he solves the problems of the lesson in competition with representatives from other schools. No pupil is permitted to participate in more than one program during the semester. This allows for seven contestants from each school. A cumulative score is kept and the school with the highest score at the end of the seventh program is the winner of the contest.

The eighth and final program in the series is the city-wide championship contest. Each school is invited to enter a seventh grade pupil in this contest. All the contestants are awarded certificates of proficiency in mental arithmetic and the city-wide champion is given a gold medal. Since each of the pupils participating in this final contest is the best in his particular school, the group, as a whole, is unusually good in mental computation. For this reason, the problems used in the contest must be carefully selected so that they are difficult enough to offer a challenge to each contestant. The three problems that follow are typical of those used in this final contest:

1. Mrs. Arnold bought 1¼ quarts of milk every day in April. She was charged 20 cents a quart. What was her milk bill for the month?
2. A can of berries that sells for 45 cents contains 2½ cups of berries. What is the cost of the berries per cup?
3. A monthly magazine sells for 25 cents a copy. A year's subscription costs $2.45. How much is saved by buying a year's subscription rather than a copy each month for a year?

Evaluation of the Program

Since this work in mental arithmetic has been in effect for two years only, it is too early to determine the improvement in mental computation that has taken

place due to this type of instruction. We do know, however, that the pupils in our seventh and eighth grade classes have become very much interested in working problems mentally. This increased interest and the practice in mental computation provided by the program should, in time, improve the entire arithmetic program of the junior high schools.

In addition to facility in the use of numbers, there are other worthwhile outcomes of the mental arithmetic program. The ability to give careful attention to what is heard or read and to select the crucial points in a discussion are valuable traits which the schools can well afford to develop. The lessons in mental arithmetic, particularly those presented over the radio cultivate these traits, since success in this subject depends upon the pupil's ability to give strict attention and to select the important points.

HERSCHEL E. GRIME
Directing Supervisor
of Mathematics
Cleveland, Ohio

Book Section

(*Continued from page 225*)

is more concise than would be suitable for any but the specialist.

From the author's viewpoint the theory of valuations is a branch of topological algebra. Historically the subject arose from the close connection between the theory of algebraic functions of one variable and the theory of algebraic numbers. Briefly, a valuation may be considered as a generalization of a norm.—D. H. POTTS, La Mesa, California.

Differential Equations (Revised Third Edition), H. B. Phillips. New York, John Wiley and Sons, 1951. viii+149 pp., $3.00.

This is an excellent text for a course in ordinary differential equations. Partial differential equations are not considered. It has an abundance of exercises, most of which are different from those of earlier editions.

Professor Phillips has included in his list of problems practically all known problems in geometry, science and engineering that can be solved by the methods of the text.

The chapter headings: 1. Differential equations of the first order with variables separable. 2. Other equations of the first order. 3. Special types of second order equations. 4. Linear equations with constant coefficients.—RALPH A. BEAVER, New York State College for Teachers, Albany, New York.

The Fourier Integral and Certain of its Applications, Norbert Wiener. New York, Dover Publications, Inc., 1951. xi+201 pp., $3.95.

This book is a republication of a famous work which first appeared in 1933. The original received many favorable reviews emphasizing such points as clearness of presentation, originality of treatment, and all-round usefulness, not only to the student but also to the engineer and the research worker. The extreme usefulness of the book and the scarcity of copies of the original certainly justifies its republication.

The material in this volume is an elaboration

(*Continued on page 236*)

WANTED: TEST ITEMS

In the article, "A Cooperative Plan for Sharing Objective-Type Test Items," in the May, 1951 issue of THE MATHEMATICS TEACHER, an appeal was made to teachers throughout the country to pool their test items and thus gain benefits from this sharing. The replies which have been received suggest that the statistical details needed have discouraged many from sending in items. In order to get this department started we would be glad to receive items, with or without the statistical data requested. It will increase the effectiveness of such items in the future if teachers could begin to collect the facts needed for the statistical summary of each, but we would like to have the actual items right now.—HENRY W. SYER, School of Education, Boston University, Boston, Massachusetts.

MATHEMATICAL MISCELLANEA

Edited by PHILLIP S. JONES
University of Michigan, Ann Arbor, Michigan

46. Superposition

The following letter and reply involve a question which, though not new, is still not uniformly well treated and hence may profitably be restated.

30622 Sheridan
GARDEN CITY, MICHIGAN
November 2, 1951

MATHEMATICS DEPARTMENT
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN

Dear Sirs:

I am a junior at the Garden City High School. During my geometry class we came across the word *superposition*. The definition given in our book was inadequate. Our teacher asked the class to find an adequate definition, but the class was unable to find one. Therefore I decided that the best place to find the definition was to write to your university. Please give me the definition you can find in your books. Thank you.

Yours sincerely,
JOSEPH J. HORZELSKI

Mathematics Department
University of Michigan
ANN ARBOR, MICHIGAN
November 15, 1951

MR. JOSEPH J. HORZELSKI
30622 SHERIDAN
GARDEN CITY, MICHIGAN

Dear Mr. Horzelski:

May I congratulate you, your teacher, and your class on your recognition of a real difficulty in geometry and upon your thoughtful and properly critical approach to it.

Dictionary definitions of *superposition* seem pretty clear if you regard them as concerned with a physical object which can be picked up and "laid upon" another physical object. However, since in geometry we are dealing with ideal or abstract figures and not physical objects, the notion of taking hold of a triangle picking it up and laying it upon another is not at all clear or sound logically. Further, a proof that two objects are congruent by superposition requires the assumption that the object superimposed does not change—i.e., that it remains the same size or congruent to itself—during the motion.

In other words, ΔABC is congruent to $\Delta A'B'C'$ by superposition only if ΔABC is congruent to $\Delta A_1B_1C_1$, $\Delta A_2B_2C_2$, etc., itself in different positions. This latter, however, can be shown true only by the superposition which we were originally trying to demonstrate to be a legitimate method of proof. This, then, is circular (hence invalid) reasoning.

In this connection Bertrand Russell, one of today's leading mathematical philosophers and logicians, says: "I may observe (a) that superposition applies to space, (b) that as a criterion of equality, it presupposes that the matter superposed is rigid, (c) that rigidity means constancy as regards metrical properties. This shows that we cannot, without a vicious circle, define spatial equality by superposition."[1]

A further objection, though not so serious, is that the notion of superposition introduces a three dimensional concept into two dimensional geometry. This objection is especially interesting when the one triangle would have to be turned over to be superimposed upon the other because the similar three dimensional problem, related to a tetrahedron, leads to considerations of four dimensional geometry.

Some people have further argued that superposition implies motion, motion involves time, and time is not a geometric concept.

Historically, this question is also interesting since some scholars believe that Euclid, himself, recognized that superposition was not clearly a geometric relationship, but that he could find no other way to prove the "side-angle-side" congruence theorem. These scholars argue thus: Euclid used superposition as rarely as possible, only when he saw no other way of proving a theorem. There were many theorems where he could have more easily used superposition, but where he did not if he saw any other proof. Hence, he used superposition unwillingly and with misgivings.

An interesting example of this is to be found in the first three theorems of Book I of Euclid.[2]

[1] Bertrand Russell, *Principles of Mathematics*. (New York: W. W. Norton & Company, Inc., 1938), p. 161.

[2] See T. L. Heath, *The Thirteen Books of Euclid's Elements*. (Cambridge University Press, 1908) for the best English translation of Euclid. Propositions 1–4 of Book I are to be found on pages 241–50, and an extended discussion of *superposition*, motion, and congruence is to be

We, today, are quite willing to use our compasses to transfer distances from one place to another in a geometric construction. Practically this is all right, but logically it too involves the notion of superposition, of picking up a length,

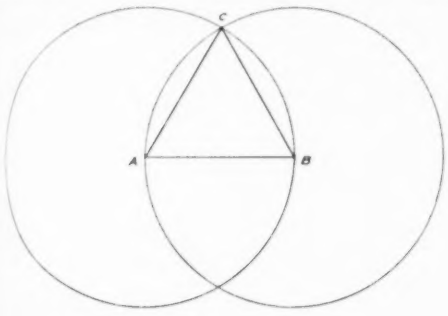


FIG. 1.

laying it down in another place, and assuming that the second distance is equal to the first. Euclid in his Proposition 2, "From a given point to draw a straight line equal to a given straight line," showed in effect that it was not necessary to think of picking up, moving, or carrying a length from one place to another.

In order to solve this problem he had in Proposition 1 to show how "To describe an equilateral triangle on a given finite straight line." The construction, in brief, was: From A, Figure 1, with radius AB draw a circle. From B with radius BA draw a circle. The circles determine C and ABC is the required triangle.

Proposition 2, "From given point A, in Figure 2, to lay off a straight line equal in

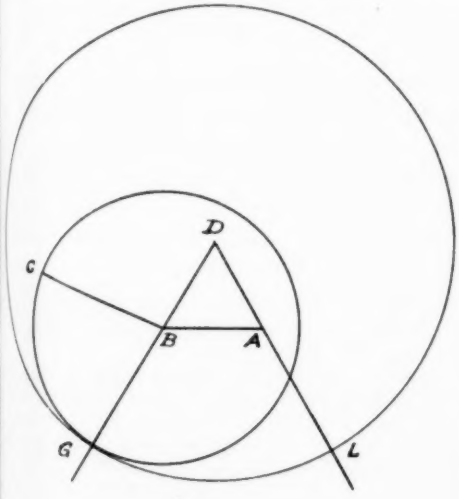


FIG. 2.

length to segment BC," is constructed as follows: Join A to B and construct equilateral triangle DAB. Extent DB and DA. Describe the circle with center at B and radius BC meeting DB extended in G. From D with radius DG draw the circle cutting DA in L. It is then easy to prove that AL is the required line through A of length equal to that of BC.

Euclid's Proposition 3 uses the above construction to solve the problem "From the greater of two given straight lines to cut off a part equal to the less" without transferring any distance with compasses or a ruler. Euclid's Proposition 4 was the S.A.S. theorem proved, however, by superposition.

So far, I have merely tried to explain further a fact which you and your classmates have already discovered for yourselves; namely, *there is no good definition of "superposition"* as it is used in geometry. I would like to congratulate you and your teacher for your having recognized this difficulty and for having attacked it intelligently and critically.

I am sure that the next question you will raise is "How, then, can we prove the S.A.S. and other theorems for which our text used superposition?" I will indicate briefly three of the different ways which are available for this.

Felix Klein, a famous German mathematician, chose to make *motion* a mathematical idea by giving it a special geometric definition. In particular, he defined two kinds of motions, *translations* and *rotations*. In brief, a *translation* was defined as a "transformation" by which every point, A, of the plane is correlated or matched with another point, A', of the plane, and every point, P, on the line joining A to A' has a mate, P', also on AA'. Further postulates about these motions guarantee that $AB=A'B'$, $\angle ABC=\angle A'B'C'$ where A, B, C are any points and A', B', C' are the "mates" or "maps" associated with them by a *motion*. *Congruent figures* are then defined to be figures which can be carried into each other by one of these mathematical (not physical) "motions."[3]

Another solution of this problem had been given earlier (1899) by the German, David Hilbert.[4] Hilbert classified the axioms necessary

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found on pages 225–31 of Volume I. An inexpensive English edition of *Euclid's Elements*, without the above notes however, is available as No. 891 of Everyman's Library sold in this country by E. P. Dutton and Co., Inc., New York.

[3] For a further explanation of this scheme see p. 162, ff. of *Elementary Mathematics from an Advanced Standpoint, Geometry*, (New York: The Macmillan Company, 1939), a translation by E. R. Hedrich and C. A. Noble of the third (1935) edition of Klein's original work which was first published in 1908. The English edition has recently been reprinted by the Dover Publishing Company.

[4] An English translation by E. J. Townsend of Hilbert's work is available as *The Foundations of Geometry*. (Chicago: The Open Court Publishing Company, 1902). Heath, *op. cit.*

for deriving Euclidean geometry as *axioms of connection, axioms of order, axiom of parallels, axioms of congruence, axiom of continuity.* His sixth axiom of congruence is essentially the S.A.S. theorem. That is, in Hilbert's system, S.A.S. can not be proven but must be assumed.

In the early part of this century there was considerable interest in analyzing carefully and completely all the assumptions necessary in Euclidean geometry. In addition to Hilbert's, sets of axioms were given by Pasch in 1882, and by Veblen, an American, in 1904. In 1932 another American, G. D. Birkhoff, gave a set of postulates which made the idea of *similarity* the basic concept.[5] Figures were defined as congruent if they were similar with a ratio one to the other of 1. He postulated then that "Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional." From these beginnings the S.A.S. congruence theorem can be proven easily.

It is not likely that in such a short space (though it may seem like a long letter to you) I have been able to make the problems related to superposition entirely clear. I have, in fact, never given you that for which you asked, a definition of *superposition*, but have instead said it can not be defined in geometry. I only hope that you will be interested in knowing that there are different ways of laying a foundation for Euclidean geometry, that there are also different kinds of geometry, and that mathematics is growing rapidly in all areas these days. I also hope you will continue to be interested in geometry as well as to be critical and careful in your geometric thinking and reading.

If I can be of any further help, please write me again.

Sincerely yours,
PHILLIP S. JONES

47. A New Approach to the Trisectrix of Maclaurin.[6]

Department Editor's Note: In both "demonstrative" and analytic geometry locus problems merit more attention and variety. It is fascinating fun to see an iterated construction give birth to a smooth curve, to derive its equation, and to explore its properties. It is for this reason and not for its relation to trisection that this "new approach" to the *Trisectrix of Maclaurin* is included here. For comparison, the "old approach" is:[7]

In Figure 3 let the point O, and the line λ at distance a from O, and the circle with $OB = 4a$ as a diameter be given. Let the line OC rotating

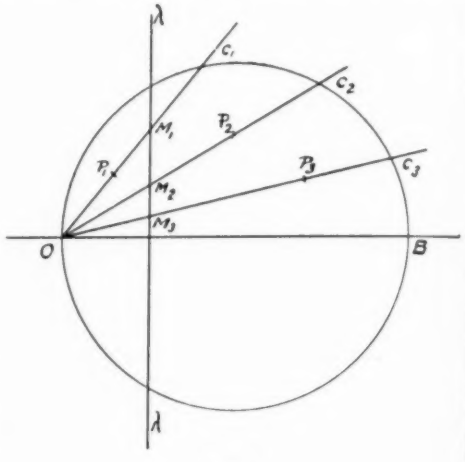

FIG. 3.

about O meet the circle in C and λ in M. The variable point P is then determined for each position of OC by laying off toward C from O, $OP = MC$. P traces the *Trisectrix of Maclaurin* whose equation with the pole at O is $r = a(4\cos\theta - \sec\theta)$. P.S.J.

This is a new approach to the construction of the *Trisectrix of Maclaurin* and its use in the trisection of an angle.

Problem: To trisect the given angle AOB or θ (Figure 4).

Construction: In Figure 4, let $a = DC$

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pp. 228–231 discusses Hilbert's system in relation to superposition as used by Euclid.

[5] G. D. Birkhoff, "A Set of Postulates for Plane Geometry, Based on Scale and Protractor." *Annals of Mathematics*, Volume 33 (April 1932), pp. 329–45. These postulates were also used in the high school text by G. D. Birkhoff and Ralph Beatley, *Basic Geometry*, (Chicago: Scott, Foresman and Company, 1941).

[6] *Department Editor's Note:* This curve, discussed by Colin Maclaurin in his *A Treatise of Fluxions* (Edinburgh: 1742) p. 260, is a special case of the "settrice di Plateau" or "curva isociclotomica," which may be used to divide an angle into any number of equal parts. This and some further data are to be found in *Enciclopedia delle Matematiche Elementari e Complemente*, Volume II, Parte II (Milano: 1950), p. 405, 406.

Percy F. Smith and Arthur Sullivan Gale's, *The Elements of Analytic Geometry*, (Ginn and Company, 1904) page 303 notes that this curve is the inverse of the hyperbola $3ax^2 - ay^2 + 2x = 0$ and R. C. Yates in his *Handbook of Curves*, (Ann Arbor: J. W. Edwards, 1947) page 163, notes that it is the first positive pedal of a parabola if the pedal point is the reflection of the focus in the directrix.

[7] Clyde E. Love, *Analytic Geometry* 3rd edition. (New York: The Macmillan Co., 1938) pp. 172–173.

$=CO$. At C erect a perpendicular to DO. Through D draw rays as shown, intersecting CG at E_1, E_2, E_3, E_4, etc. With O as center and OE_1, OE_2, OE_3, OE_4, etc., as radii, describe arcs intersecting the prolongations of the rays at F_1, F_2, F_3, F_4, etc.,

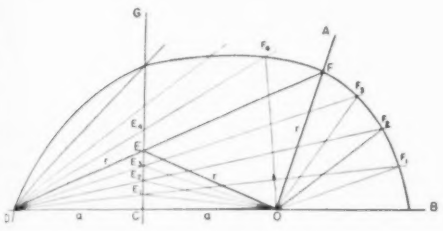

FIG. 4.

respectively. Draw a smooth curve (locus) through F_1, F_2, F_3, F_4, etc. Let this locus intersect OA at F. Draw DF. Angle FDO is equal to $\frac{1}{3}$ of angle AOB.

Proof: Triangle DEO is an isosceles triangle since $DE=EO$. Hence $\angle EDO=\angle EOD$. Then $\angle FEO=2\angle FDO$ since it is an exterior angle of triangle DEO. Triangle FOE is also isosceles since $EO=OF$. Hence $\angle FED=\angle EFO$.

$$\angle AOB=\angle FDO+\angle DFO$$
$$=\angle FDO+2\angle FDO.$$

Therefore $\angle FDO=\frac{1}{3}\angle AOB$.

Figure 5 shows the complete curve. It will be observed that the equation of the curve referred to O as pole and OB as polar axis is:

$$r=\frac{a}{\cos\frac{\theta}{3}}, \text{ where } DE=EO=EF=r;$$

$$\text{angle } EDO=\frac{\theta}{3} \text{ and angle } FOB=\theta.$$

The "old" equation of the *Trisectrix of Maclaurin*, which is referred to point D as pole and DB as the polar axis, may be derived from the method of development suggested here as follows:

In the $\triangle FDO$, let $\angle FDO=\theta$, then by the cosine law

$$r^2=4a^2+\frac{a^2}{\cos^2\theta}$$
$$-2\cdot 2a\left(\frac{a}{\cos\theta}\right)[\cos(180-3\theta)]$$
$$=4a^2+\frac{a^2}{\cos^2\theta}+\frac{4a^2(\cos 3\theta)}{\cos\theta}$$

but $\cos 3\theta=\cos^3\theta-3\sin^2\theta\cos\theta$

therefore

$$r^2=4a^2+\frac{a^2}{\cos^2\theta}+\frac{4a^2(\cos^3\theta-3\sin^2\theta\cos\theta)}{\cos\theta}$$
$$=4a^2+a^2\sec^2\theta+4a^2\cos^2\theta-12a^2\sin^2\theta$$
$$=a^2[4+\sec^2\theta+4\cos^2\theta-12(1-\cos^2\theta)]$$
$$=a^2(4+\sec^2\theta+4\cos^2\theta-12+12\cos^2\theta)$$
$$=a^2(16\cos^2\theta-8+\sec^2\theta)$$
$$=a^2(4\cos\theta-\sec\theta)^2$$
$$r=a(4\cos\theta-\sec\theta)$$

which is the desired "old" form of the polar equation of the *Trisectrix of Maclaurin*.

My fellow teachers have found the method suggested in this work to be interesting, and in my analytics class it has proved to be enlightening to the students. I hope that this humble work will contribute towards a greater appreciation of geometry, the first and noblest of science.

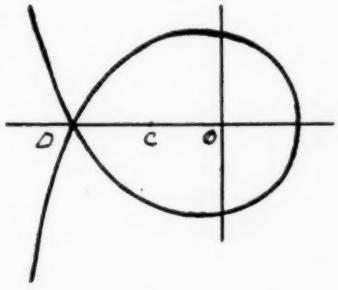

FIG. 5.

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48. The Science Talent Search

We discussed a year or so ago in this department the opportunities for mathematics teachers and students offered by the Westinghouse Science Talent Search.[8] It seems worthwhile, however, to report progress and to remind our readers of this continuing opportunity.

"Miscellanea 37. Still More About Nedians" included part of a thesis written by Marilyn R. Taig for the 1950 science contest. The pamphlet, *How You Can Search for Science Talent*, recently distributed by Science Clubs of America to announce the eleventh annual "Search," lists the forty Winners' of the tenth "Search." Six winners had written mathematical reports.

[8] THE MATHEMATICS TEACHER, Vol. XLIII (January 1950), p. 35.

Professor P. D. Edwards of Ball State Teachers College, Muncie, Indiana has written us about the Indiana Science Talent Search, sponsored jointly by the Indiana Academy of Science and the Indianapolis *Times*. Last year, as a member of a committee of the Academy, he read the 85 essays which had been submitted, of which eight were on mathematics. Of the eight, four displayed an interest and understanding of mathematics far beyond the content of the course of study. These four were among the group invited to come to Indianapolis for personal interviews with the committee. This shows the way in which the stimulus and recognition of the Science Talent Search can spread out into a local area. Even the four whose mathematical papers were less outstanding, had received a stimulus, achievement, and recognition which is invaluable.

Book Section

(*Continued from page 231*)

of a series of lectures which Dr. Wiener gave at Cambridge in 1932. His purpose was to present the theory of the Fourier integral using some of the modern tools of analysis, in particular, Lebesgue integration. This latter subject is not assumed as a prerequisite, a brief summary of the necessary results being contained in the book. The subject matter should be accessible to the first-year graduate student who has already been well grounded in the "pre-Lebesgue" theory of real variables.—D. H. POTTS, La Mesa, California.

Introduction to the Theory of Algebraic Functions of One Variable, Claude Chevalley. New York, American Mathematical Society, 1951. xii + 188 pp., $4.00.

This volume is Number 6 in the Mathematical Surveys Series of the American Mathematical Society. The subject matter of the book is indicated by its title. But it must be added that, in spite of the presence of the word "introduction" in the title, the book is intended for mathematicians of a rather advanced level. Such topics as abstract algebra, functions of a complex variable, and topology are assumed as prerequisites.

y is an algebraic function of x if it satisfies an equation of the form $F(x, y) = 0$ where F is a polynomial whose coefficients are elements of an arbitrary field. A finite field of constants leads one to the theory of algebraic numbers, if the field is that of the complex numbers one obtains from the standard algebraic functions of a complex variable.—D. H. POTTS, La Mesa, California.

Illinois State Normal University will hold its fifth annual **Spring Conference** on the teaching of elementary and secondary school mathematics on Saturday, March 29, 1952. The theme for the Conference will be "A Functional Approach to the Teaching of Mathematics" and the principal speakers will be Dr. Howard Fehr of Teachers College, Columbia University and Miss Anne Gustafson, Supervisor of Elementary Instruction at Rockford, Illinois. These speakers will also participate in a panel discussion at the afternoon session together with Dr. Kenneth Henderson of the University of Illinois and Dr. Bjarne Ullsvik of Illinois State Normal University.

The Second Section Conference of the **Illinois Council of Teachers of Mathematics** will be held on the campus of the **Western Illinois State College** under the sponsorship of the college and the public schools of Macomb on Saturday, April 26, 1952. Dr. Henry Van Engen, Head, Department of Mathematics, Iowa State Teachers College will address the conference on "Some Thoughts on Problem Solving." Dr. John R. Mayor, Professor of Mathematics and Education, University of Wisconsin will speak on "Key Concepts in Algebra."

In addition to the two keynote speakers, more than twenty classroom teachers will participate in directing discussion on a number of current topics on mathematics. There will also be a panel discussion on both the elementary and secondary level.